# Chapter 1 Set Theory and Logic



## 1 Fundamental Concepts

Exercises:

(1) Trivial.

(2)
(a) Only "$\Rightarrow$". (b) Only "$\Rightarrow$". (c) True.
(d) Only "$\Leftarrow$". (e) True. (f) Only "$\subseteq$".
(g) True. (h) False. (i) True.
(j) True. (k) False. (l) True.
(m) Only "$\subseteq$". (n) True. (o) ~~True~~. Only "$\subseteq$".
(p) True. (q) True.

(3) Trivial.

(4) Trivial.

(5) Trivial.

(6) Trivial.

(7) Trivial.

(8) Trivial.

(9) Trivial.

(10)
(a) $\{(x,y) \mid x \text{ is an integer}\} = \mathbb{Z} \times \mathbb{R}$
(b) $\{(x,y) \mid 0 < y \leq 1\} = \mathbb{R} \times (0,1]$
(c) Not a cartesian product.
(d) $\{(x,y) \mid x \text{ is not an integer}, y \text{ is an integer}\} = (\mathbb{R} \setminus \mathbb{Z}) \times \cancel{\mathbb{R}}\,\mathbb{Z}$
(e) Not a cartesian product.

## 2 Functions

Exercises:

(1) (a) *Proof.* Let $f: A \to B$ be a function and let $A_0 \subseteq A$. ~~and $B_0 \subseteq B$~~. Let $x \in A_0$ and set $y = f(x)$. Then $y \in f(A_0)$ and, by definition, $x \in f^{-1}(f(A_0))$. Hence, ~~$A_0 \subseteq$~~ $A_0 \subseteq f^{-1}(f(A_0))$. Now suppose that $f$ is injective. ~~Let $x \in f^{-1}(f(A_0))$. Then there is a unique $y \in B$ with $f(x) = y$, $f^{-1}(y) = x$. Thus, $y \in f(A_0)$ and as $y$ is unique for~~ Let $x \in f^{-1}(f(A_0))$. Then there is some $y \in f(A_0)$ with $f(x) = y$. Moreover, there is a $\tilde{x} \in A_0$ such that $f(\tilde{x}) = y$ and as $f$ is injective, $\tilde{x} = x$. $\square$

(b) Analogous.

(2) Easy.

(3) Trivial.

(4) Repetitive.

(5) Repetitive.

(6) ???

## 3 Relations

Definition (Immediate ~~Pred~~ Predecessor/Successor): Let $X$ be a set and let $<$ be an order. Then, for $a, b \in X$,

$$(a,b) := \{ x \in X \mid a < x < b \}$$

denotes ~~the~~ an <u>open interval</u> in $X$. If $(a,b) = \emptyset$, ~~then~~ $a$ is the <u>immediate predecessor</u> of $b$, and $b$ is the <u>immediate successor</u> of $a$.

Definition (Order Type): Let $A$, $B$ be sets with order relations $<_A$, $<_B$, respectively. We say $A$ and $B$ have the same <u>order type</u> if there is a bijection $f: A \to B$ such that

$$a_1 <_A a_2 \implies f(a_1) <_B f(a_2)$$

for all $a_1, a_2 \in A$.

**Definition** (Least Upper Bound Property): An ordered set $X$ is said to have the <u>least upper bound property</u> if every nonempty subset $A \subseteq X$ that is bounded from above has a least upper bound (supremum).

Exercises:

(1) Proof.

(i) Reflexivity. Let $(x, y) \in \mathbb{R}^2$. Then $x - y^2 = x - y^2$, so $(x, y) \sim (x, y)$.

(ii) Symmetry. Let $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ with ~~(x_0, y~~ $(x_0, y_0) \sim (x_1, y_1)$. Then $x_0 - y_0^2 = x_1 - y_1^2$, so ~~clear~~ clearly $x_1 - y_1^2 = x_0 - y_0^2$, too. Hence, $(x_1, y_1) \sim (x_0, y_0)$.

(iii) Transitivity. Let $(x_0, y_0), (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ with $(x_0, y_0) \sim (x_1, y_1)$ and $(x_1, y_1) \sim (x_2, y_2)$. Then

$$x_0 - y_0^2 = x_1 - y_1^2 \quad \text{and} \quad x_1 - y_1^2 = x_2 - y_2^2,$$

so $(x_0, y_0) \sim (x_2, y_2)$, too. $\square$

(2) Trivial.

(3) We ~~don't~~ do not have that for any $a$, there is a $b$ with $aCb$. For instance, $C = \emptyset$ over $\mathbb{R}$ is symmetric and transitive, but not reflexive.

(4) Trivial.

~~(5) (a) Proof.~~

(5) Easy.

(6) Trivial.

(7) Trivial.

(8) Trivial.

(9) Trivial.

(10) Easy.

(11) Proof. Let $X$ be an ordered set and let $a \in X$. Suppose $a$ has two immediate ~~predecessors $b, b' \in X$. Then $(a,b) = (a,b') = \emptyset$, so neither $a < b'$~~ successors $b, b' \in X$. Thus, $(a,b) = (a,b') = \emptyset$ while $a < b, b'$. Hence, neither $b < b'$ nor $b' < b$ as then $b \in (a,b')$ or ~~$b' \in (a,b')$~~ $b' \in (a,b)$, respectively. Thus, $b = b'$. Analogous for the immediate predecessor of $a$. $\square$

Min./Max. unique is trivial.

(12) Easy.

(13) ~~Easy~~ Repetitive.

~~(14) (a) Proof. "If:" Suppose $C = D$. Let $a, b \in A$ such that $aDb$. That is, $(b,a) \in C$. As $C = D$, also $bDa$. "Only if:" Suppose~~

(14) (a) Proof. "If:" Suppose $C = D$; let $a, b \in A$ such that $aCb$. As $C = D$, also $aDb$. Thus, by def., $(b,a) \in C$, i.e., $bCa$. "Only if:" Suppose $C$ is symmetric. Let $(a,b) \in C$, then also $(b,a) \in C$, so $(a,b) \in D$. Let $(a,b) \in D$, then ~~also~~ $(b,a) \in C$, so $(a,b) \in C$. $\square$

(b) Trivial.

(c) ~~Ea~~ Repetitive.

(15) Easy.

4 <u>The Integers and the Real Numbers</u>

Exercises:

(1) Repetitive.

(2) Repetitive.

(3) (a) Proof. Let ~~$\mathcal{A}$~~ ~~$\mathcal{A}$~~ $\mathcal{A}$ be a collection of inductive sets. Set $D = \bigcap_{A \in \mathcal{A}} A$. Clearly $1 \in D$ as $1 \in A$ for all $A \in \mathcal{A}$. Let $x \in D$. Then $x \in A$ for all $A \in \mathcal{A}$. Thus, $x + 1 \in A$ for all $A \in \mathcal{A}$ and therefore $x + 1 \in D$. $\square$

(b) ~~Easy?~~ → next page

Definition (Inductive Set): A subset $A \subseteq \mathbb{R}$ is <u>inductive</u> if $1 \in A$ and for all $x \in A$, also $x+1 \in A$.

Definition (Integers): Let $\mathcal{A}$ be the collection of all inductive subsets of $\mathbb{R}$. Then the set $\mathbb{Z}_+$ of <u>positive integers</u> is

$$\mathbb{Z}_+ = \bigcap_{A \in \mathcal{A}} A.$$

The integers $\mathbb{Z}$ are defined to be $\mathbb{Z} = \mathbb{Z}_+ \cup (-\mathbb{Z}_+) \cup \{0\}$. If $n \in \mathbb{Z}_+$ is a positive integer, we define $S_n$ to be the <u>section</u> of positive integers less than $n$, i.e.,

$$S_{n+1} = \{1, 2, \dots, n\}.$$

(3) (b) (1) Follows readily from part (a).

(2) Proof. Let $A$ be an inductive set of positive integers. Clearly, $\mathbb{Z}_+ \subseteq A$ as $A \in \mathcal{A}$ where $\mathcal{A}$ is the ~~set of~~ collection of all inductive subsets of $\mathbb{R}$ and $\mathbb{Z}_+ = \bigcap_{B \in \mathcal{A}} B$. ~~For $A \subseteq \mathbb{Z}_+$, let $n \in A$.~~ $A \subseteq \mathbb{Z}_+$ is trivial as $A$ is defined as such. $\square$

(4) (a) Proof. Let $n=1$. Clearly, $\{1\}$ has a largest element, $1$. Suppose that the ~~same~~ property holds for some ~~$n \in$~~ $n \in \mathbb{Z}_+$, i.e., that every subset of $\{1, \dots, n\}$ has a largest element. Consider $\{1, \dots, n+1\}$. Let $C \subseteq \{1, \dots, n+1\}$. If $n+1 \in C$, then $n+1$ is the largest element. ~~Otherwise,~~

$$\cancel{C \cap \{1, \dots, n\} \subseteq \{1, \dots, n\}},$$

~~so Oth~~ Otherwise, $C \subseteq \{1, \dots, n\}$, and $C$ has a largest element by the induction hypothesis. $\square$

~~(b) Part (a) only covers finite subsets subsets~~

(b) Part (a) only covers finite subsets of $\mathbb{Z}_+$, but arbitrary subsets may be infinite.

(5) (a) Proof. Let $a \in \mathbb{Z}_+$ and consider

$$X = \{ x \mid x \in \mathbb{R},\ \cancel{x+a}\ a+x \in \mathbb{Z}_+ \}.$$

Clearly, $1 \in X$ as $a+1 \in \mathbb{Z}_+$ due to $\mathbb{Z}_+$ being inductive. Let $x \in X$. Then $\cancel{x+a}\ a+x \in \mathbb{Z}_+$ and therefore $(a+x)+1 \in \mathbb{Z}_+$. Hence, $x+1 \in X$, too. Thus, $X$ is inductive. ~~and~~ ~~Let $b=1$, then~~ ~~clearly~~ Hence, as $\mathbb{Z}_+ \subseteq X$, we have $a+b \in \mathbb{Z}_+$ for all $a, b \in \mathbb{Z}_+$.

$\square$

~~(b) Follows~~

(b) Proof. Let $a \in \mathbb{Z}_+$ and consider

$$X = \{ x \mid a x \in \mathbb{Z}_+ \}.$$

Clearly $1 \in X$. Let $b \in X$. Then

$$a(b+1) = \underbrace{a b}_{\in \mathbb{Z}_+} + b \in \mathbb{Z}_+$$

by (a).

$\square$

(c) Proof. ~~Let $a \in \mathbb{Z}_+$ and~~ ~~c~~Consider

$$X = \{ x \mid x \in \mathbb{R},\ x-1 \in \mathbb{Z}_+ \cup \{0\} \}.$$

Clearly $1 \in X$ as $1-1=0 \in \mathbb{Z}_+ \cup \{0\}$. Let $a \in X$. Then $(a+1)-1 = a \in \mathbb{Z}_+ \cup \{0\}$, so $a+1 \in X$. Hence, $X$ is inductive and $\mathbb{Z}_+ \subseteq X$.

$\square$

(d) Repetitive.

(e) Repetitive.

(6) Easy.

(7) Easy.

~~(8) (a) Proof. Let $A \subseteq \mathbb{R}$ be bounded from above.~~

(8) (a) Proof. Let $A \subseteq \mathbb{R}$ be bounded from below. Then the set $-A := \{-x \mid x \in A\}$ ~~has~~ is bounded from above and has a least upper bound $u$. That is, for all $-x \in -A$, we have $-x \le u$, i.e., $-u \le x$. Hence, $-u$ is a lower bound for $A$. Let $\epsilon > 0$. Then there is some $-x \in -A$ such that $-x > u - \epsilon$. That is, $-u+\epsilon > x$, so $-u$ is the least lower bound of $A$.

$\square$

(8) (b) *Proof.* Consider $\{1/n \mid n \in \mathbb{Z}_+\}$. Suppose there is some $l>0$ such that $l$ is a lower bound, i.e., a lower bound greater than $0$. However, ~~b~~ by the Archimedean property, we may choose an $n \in \mathbb{Z}_+$ with $n > 1/l$. Thus, $l > n$. ↯ Hence, $l$ is not a lower bound and we find that the infimum is $0$. (It is clear that $0$ is a lower bound.) $\square$

(c) ~~Rep~~ Repetitive.

(9) Repetitive.

(10) (a) *Proof.* Let ~~$x<0$~~ $x>0$ and $0 \leq h < 1$. Then

$$\begin{aligned}(x+h)^2 &= x^2 + h^2 + 2hx \\ &\leq x^2 + h + 2hx \\ &= x^2 + h(2x+1);\end{aligned}$$

$$\begin{aligned}(x-h)^2 &= x^2 + h^2 - 2hx \\ &\geq x^2 - 2hx \\ &= x^2 - h(2x).\end{aligned}$$

$\square$

~~(b) *Proof.* Let $x>0$, and suppose that $x^2<a$. Let $0<b<a^2$ and set $h = \tilde{h}/(2x+1)$ with $\tilde{h}=$~~

(b) *Proof.* Let $x>0$. Suppose $x^2<a$. Choose $\tilde{h} < a - x^2$ ~~and set $h=\tilde{h}/(2x+$~~ such that $\tilde{h}<1$ and set $h = \tilde{h}/(2x+1)$. Clearly ~~$0<h\leq 1$~~ $0 \leq h < 1$. Thus,

$$(x+h)^2 \leq x^2 + h(2x+1) = x^2 + \tilde{h} < a.$$

Analogous for $(x-h)^2 > a$ ~~f~~ for some $h>0$ if $x^2 > a$. $\square$

~~(10) (c) Proof. Let $a>0$. Define~~

$$\cancel{B = \{ x \in \mathbb{R} \mid x^2 < a \}}$$

~~and let $0<b<a$. $0<b<\min\{1, a\}$. Then $b^2<b<a$, so $b \in B$, where $b$ is positive. Suppose that $a>1$. Then $a^2>a$. Let $b = \sup B$. We want to show that $b^2=a$. $b^2=a$~~

(10) (c) TODO

~~(d) Proof. Let $b,c>0$ and suppose that $b^2=c^2$.~~

(d) Proof. Let $b,c>0$. Suppose $b \neq c$ (w.l.o.g., $b<c$) and set $\tau = c-b$. Then $\tau>0$, $c = b+\tau$. Hence,

$$c^2 = (b+\tau)^2 = b^2 + \tau^2 + 2b\tau > b^2,$$

i.e., $b^2 \neq c^2$.

$\square$

(11) Trivial.

## 5 Cartesian Products

Exercises:

(1) Trivial.

(2) Trivial.

(3) Trivial.

(4) Easy/Repetitive.

(5) (a) $\{ x \mid x_i \text{ is an integer for all } i \} = \mathbb{Z}^\omega$

(b) $\{ x \mid x_i \geq i \text{ for all } i \} = \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 2} \times \cdots$

(c) $\{ x \mid x_i \text{ is an integer for all } i \geq 100 \} = \mathbb{R}^{99} \times \mathbb{Z}^\omega$

(d) Not expressable as a cartesian product.

## 6 Finite Sets

Exercises:

(1) Trivial.

(2) Contraposition of Corollary 6.6.

(3) $f : X^{\omega} \to X^{\omega} : (x_1, x_2, \ldots) \mapsto (1 - x_1, 1 - x_2, \ldots)$; $X = \{0, 1\}$

~~(4) Proof. Let $x, \tilde{x} \in X^{\omega}$ such that $f(x) = f(\tilde{x})$. Then, for all $i \in \mathbb{N}$, we have $1 - x_i = 1 - \tilde{x}_i$ and therefore $x_i = \tilde{x}_i$. That is, $x = \tilde{x}$. Hence, $f$ is injective.~~

Proof. Let $x \in X^{\omega}$ and $i \in \mathbb{N}$. Then $(f(x))_i = 1 - x_i$ and $(f(f(x)))_i = 1 - (1 - x_i) = x_i$, so $f(f(x)) = x$ and $f = f^{-1}$. As ~~$f$ has an inverse, it is bijective.~~ $f$ is its own inverse, it is bijective. $\square$

(4) Easy.

(5) No, consider $A = \mathbb{R}$, $B = \emptyset$, then $A \times B = \emptyset$ is finite.

(6) Easy.

(7) Trivial.

## 7 Countable and Uncountable Sets

Exercises:

(1) Repetitive.

(2) Repetitive

(3) Repetitive.

(4) Repetitive.

(5) (a) Countable, it's just $\{\{(0, a), (1, b)\} \mid a, b \in \mathbb{Z}_+\}$.

(b) Countable, ~~same~~ same argument.

(c) Countable, union over countably many countables.

~~(d) Countable; "glorified" $\mathbb{R}$ $\mathbb{Z}_+ \times \mathbb{Z}_+$.~~

~~(e)~~ (d) Uncountable; these are all sequences.

(e) Uncountable; all sequences of 0's and 1's.

(5) (f) Countable.

(g) Countable.

(h) Countable.

(i) Countable.

(j) Uncountable.

(6) TODO

(7) Repetitive.

(8) TODO

(9) Unrelated?

## 8 The Principles of Recursive Definition

Skipped.

## 9 Infinite Sets and the Axiom of Choice

Axiom of Choice: Let $\mathcal{A}$ be a collection of ~~dis~~ disjoint non-empty sets. Then there is a set $C$ such that $C$ contains exactly one ~~ell~~ element of each $A \in \mathcal{A}$. That is,

$$C \subseteq \bigcup_{A \in \mathcal{A}} A \qquad \text{and} \qquad C \cap A = \{x\}$$

for some $x \in A$ and all $A \in \mathcal{A}$. We can say that the set $C$ <u>chooses</u> one element of each set in $\mathcal{A}$.

Exercises:

(1) Proof. Let $n \in \mathbb{Z}_+$ be arbitrary and consider its binary expansion $n_0, n_1, \dots, n_K \in \{0, 1\}$ with $n_0$ being the least significant bit. That is,

$$z = n_0 2^0 + n_1 2^1 + \dots + n_K 2^K.$$

Set $x = (n_0, n_1, \dots, n_K, 0, 0, \dots)$. Clearly $x \in X^\omega$ where $X = \{0, 1\}$. We use this procedure to define a function $\cancel{f : \mathbb{Z}_+ \to X^\omega}$ $f : \mathbb{Z}_+ \to X^\omega$. As for ~~each~~ all $n, n' \in \mathbb{Z}_+$ their binary decomposition is unique, $f$ is injective.

$\square$

(2) (a) As each (nonempty) $A \subseteq \mathbb{Z}_+$ has a well-defined minimum, we can define a choice function as

$$c : \mathcal{A} \to \mathbb{Z}_+ : A \mapsto \min A.$$

(b) For each $B \in \mathcal{B}$, we can choose the $n \in B$ such that $|n| \leq |m|$ for all $m \in B$, breaking ties by choosing the positive. That is, we choose the element closest to zero by ~~usta~~ checking whether $0, 1, -1, 2, -2, \ldots$ are in $B$.

(c) Not possible without the axiom of choice as $\mathbb{Q}$ fulfills the continuum hypothesis.

(d) Also not possible without the axiom of choice.

(3) Proof. Let $c : 2^A_{\mathcal{A}} \setminus \{\emptyset\} \to A$ be a choice function. We want to define an injection $f : \mathbb{Z}_+ \to A$. We define $f$ as:

$$f(1) = f_1(1)$$

$$f(i) = c(f_i(\{1, \ldots, i\}) \setminus f(\{1, \ldots, i-1\}))$$

As $|f_i(\{1, \ldots, i\})| = i$ and $|f(\{1, \ldots, i-1\})| = i-1$, the set is nonempty so a choice can be made. Clearly $f$ is injective, so $A$ is infinite. $\square$

We cannot define $f$ without the axiom of choice.

(4) Proof of Theorem 7.5.

(5) (a) Proof. Let $f : A \to B$ be surjective. That is, for all $b \in B$ there is some nonempty set $A_0 \subseteq A$ such that $f(a) = b$ for all $a \in A_0$. ~~Let $c : 2^A \to A$ be a choice function~~ Let $c : 2^A \setminus \{\emptyset\}^{\mathcal{U}} \to A$ be a choice function and define $h : B \to A$ by $h(b) = c(f^{-1}(\{b\}))$. As the ~~prea~~ preimage is nonempty, $h$ is well-defined. Let $b \in B$, then

$$(f \circ h)(b) = f(h(b)) = f(c(f^{-1}(\{b\}))) = b,$$

so $h$ ~~is $f$'s~~ is a right inverse of $f$. $\square$

~~(b) Proof. Let $f : A \to B$ be injective, let $\mathcal{B}$ be the collecto~~

(5) (b) Proof. Let $f: A \to B$ be injective with $A \neq \emptyset$. Let $b \in \operatorname{Im} f$, then there is exactly one $a \in A$ with $f(a) = b$. Denote this element by $f^{-1}(b)$. Then we can regard $f^{-1}$ as a function $f^{-1}: \operatorname{Im} f \to A$ such that $(f^{-1} \circ f)(a) = f^{-1}(f(a)) = a$ for all $a \in A$, so $f^{-1}$ is a left inverse. The axiom ~~of~~ of choice is not needed as $f^{-1}(\{b\})$ contains ~~a~~ at most one element so ~~no~~ no arbitrary choice has to be made.

$\square$

~~(6) (a) Let A be the set of all sets. Clearly $\mathcal{P}(A)$ is a set, so $\mathcal{P}(A$ $\mathcal{P}(A) \subseteq$~~

(6) (a) Let $\mathcal{A}$ be the set of all sets. As $\mathcal{P}(\mathcal{A})$ is a set of sets, clearly $\mathcal{P}(\mathcal{A}) \subseteq \mathcal{A}$. ~~However, $f: \mathcal{P}(\mathcal{A}) \to \mathcal{A}$ given by $f(A) = A$ is an bijection which contradicts Theorem 7.8 as as~~

(b) If $\mathcal{B} \notin \mathcal{B}$, then $\mathcal{B} \in \mathcal{B}$; but if $\mathcal{B} \in \mathcal{B}$, then $\mathcal{B} \notin \mathcal{B}$.

(7) (a) Proof. Let $A$ be uncountable. ~~By Theorem~~ As $A$ is infinite, there is an injection $f: \mathbb{Z}_+ \to A$. However, there cannot be an ~~injectio~~ injection $g: A \to \mathbb{Z}_+$ as $A$ is uncountable. Thus, $A$ has a greater cardinality that $\mathbb{Z}_+$.

$\square$

(b) Proof. Let $A, B$ and $C$ be sets such that $A$ has greater cardinality than $B$ and $B$ has greater cardinality than $C$. Let $\cancel{f: A \to B}$ $f: B \to A$ and $\cancel{f: C}$ $g: C \to B$ be injections. Let $a \in A$ and $b, b' \in B$ such that $f(b) = f(b') = a$, then $b = b'$. Let $c, c' \in C$ such that $g(c) = g(c') = b$, then $c = c'$. Thus, $\cancel{g \circ f}$ $\cancel{f \circ g}$ $f \circ g: C \to A$ is an injection. Now suppose there ~~where~~ were an injection $h: A \to C$, then $g \circ h: A \to B$ and $h \circ f: \cancel{A} B \to C$ would be injections. $\lightning$ Hence, there is no injection from $A$ to $C$ and thus $A$ has greater cardinality than $C$.

$\square$

(c) Set $A_1 = \cancel{\mathbb{Z}_+}\ \{0\}$ and $A_{n+1} = \mathcal{P}(A_n)$. It follows from Theorem 7.8 that this sequence is of increasing cardinality.

(d) The set $\mathbb{Z}_+$ has greater cardinality as every $A_n$ as all $A_i$'s are finite.

(8) TODO

10 Well-Ordered Sets

Definition (Well-Ordered Set): A set $A$ with order relation $<$ is well-ordered if every nonempty subset of $A$ has a smallest element.

Theorem (Well-Ordering): Let $A$ be a set. Then there exists an order relation on $A$ that is a well-ordering.

Corollary: There exists an uncountable well-ordered set.

Definition (Section): Let $X$ be a well-ordered set. Let $\alpha \in X$, then the set

$$S_\alpha := \{ x \in X \mid x < \alpha \}$$

is called the section of $X$ by $\alpha$.

Lemma (Minimal Uncountable Well-Ordered Set): There exists a well-ordered set $A$ having a largest element $\Omega$ such that the section $S_\Omega$ of $A$ by $\Omega$ is uncountable but every other section of $A$ is countable.

Theorem: If $A \subseteq S_\Omega$ is ~~coun~~ countable, then $A$ has an upper bound.

Exercises:

(1) Proof. Let $X$ be well-ordered and let $A \subseteq X$ be a nonempty subset bounded from above. That is, the set

$$\{ x \in X \mid a \leq x \text{ for all } a \in A \}$$

is nonempty. As $X$ is well-ordered, then this set has a minimal element $u$ which is the least upper bound of $A$ in $X$. □

(2) (a) Proof. Let $X$ be well-ordered and let $a \in X$ not be the largest element. Set $A = \{ x \in X \mid a < x \}$, then $A \neq \emptyset$ as $a$ is not the largest element and $A$ has a ~~maximum~~ minimum $b \in A$. ~~such that~~ Thus, $(a, b) = \emptyset$ and $b$ is ~~$\alpha$~~ $a$'s immediate successor. □

(b) The integers $\mathbb{Z}$.

(3) No, $\{1,2\} \times \mathbb{Z}_+$ and $\mathbb{Z}_+ \times \{1,2\}$ do not have the ~~same~~ same order type.

Proof. Suppose there is a bijection ~~$f: \{1,2\} \times$~~

$$f: \{1,2\} \times \mathbb{Z}_+ \to \mathbb{Z}_+ \times \{1,2\}$$

such that for all $a, b \in \{1,2\} \times \mathbb{Z}_+$ with $a < b$ we have $f(a) < f(b)$.

TODO

(4) (a) Proof. Let $A$ be an ordered set. "If:" Suppose there is some $A_0 \subseteq A$ such that there is a ~~bijection $f: A_0 \to \mathbb{Z}_+$ with that is order-preserving.~~ bijection $f: \mathbb{Z}_- \to A_0$ that is order-preserving. Let $C \subseteq \mathbb{Z}_-$ be a subset with no minimum. That is, for all $x \in C$ there is a $x_0 \in C$ ~~not~~ with $x_0 < x$. But then also $f(x_0) < f(x)$, so ~~Im~~ $f(C)$ also has no minimum and thus $A_0$ is not well-ordered. Thus, $A$ is also not well-ordered. "Only if:" Suppose $A$ is not well-ordered. Clearly, $A$ has to be infinite. ~~and there is an injection $f: \mathbb{Z}_+ \to A$.~~ Let $c: \mathcal{A} \to A$ be a choice function where $\mathcal{A}$ is the set of all nonempty subsets of $A$. We define ~~$f: \mathbb{Z}_+$ $g: \mathbb{Z}_- \to A$ as follows:~~

$\cancel{g(-1) = c(A)}$

$\cancel{g(-n) = c}$

$g: \mathbb{Z}_+ \to A$ as follows:

$$g(1) = c(A)$$

$$g(n) = c(\{a \in A \mid a < b \text{ for all } b \in g(\{1, \dots, n-1\})\})$$

~~And set $f: \mathbb{Z}_- \to A: -n \mapsto g(n)$. Clearly $f$ is~~
We set $f: \mathbb{Z}_- \to \operatorname{Im} g: -n \mapsto g(n)$. Clearly $f$ is ~~an~~ bijective and order-preserving.

□

(b) Proof. Let $A$ be ~~well-order~~ ordered. Suppose $A$ is not well-ordered. ~~Then there is a bijection $f: \mathbb{Z}_- \to A$~~ Then ~~A~~ some $A_0 \subseteq A$ and $\mathbb{Z}_-$ have the same order type and thus $A_0$ ~~on~~ is countable and not well-ordered. By contraposition, the statement follows.

□

(5) Proof. ~~Supp~~ Suppose that the well-ordering theorem holds. To show the axiom of choice, let $\mathcal{A}$ be a collection of disjoint nonempty sets and set

$$\tilde{A} = \bigcup_{A \in \mathcal{A}} A.$$

By the well-ordering theorem, there is an order $<$ over $\tilde{A}$ such that $\tilde{A}$ is ~~ordere~~ well-ordered. We can now set

$$\cancel{C = \{} \qquad C = \bigcup_{A \in \mathcal{A}} \{\min A\}.$$

Clearly, $|C \cap A| = 1$ for all $A \in \mathcal{A}$, so the axiom of choice holds.

□

(6) (a) Proof. ~~Supp~~ Suppose $S_\Omega$ has a largest element, ~~say~~ say $u$. Then $x \leq u$ for all $S_\Omega$ and

$$S_\Omega = S_u \cup \{u\}. \lightning$$

~~However, $S_\Omega$ is uncountable and~~ $S_u$ is countable. This contradicts that $S_\Omega$ is uncountable.

□

(b) Proof. Let $\alpha \in S_\Omega$. Then

$$S_\Omega = S_\alpha \cup \{\alpha\} \cup \{x \in S_\Omega \mid \alpha < x\}.$$

As both $S_\alpha$ and $\{\alpha\}$ are countable and $S_\Omega$ is ~~un~~ uncountable, the ~~r~~ rightmost set must be uncountable.

□

~~(c) Proof. We want to show that~~

$$\cancel{X_0 = \{x \in S_\Omega \mid (x,y) \neq \emptyset \text{ for all } y \in S_\Omega\}}$$

~~is uncountable. We can also show that~~

$$\cancel{X_0^c = \{x \in S_\Omega \mid (x,y) = \emptyset \text{ for some } y \in S_\Omega\}}$$

~~is countable.~~

(6) (c) Proof. We want to show that

$$X_0 = \{ b \in S_\Omega \mid (a,b) \neq \emptyset \text{ for all } a \in S_\Omega \}$$

is uncountable. We show that $X_0$ has no upper bound in $S_\Omega$. ~~Supp~~ Suppose $u \in S_\Omega$ is an upper bound of $X_0$, i.e., $b \leq u$ for all $b \in X_0$.

TODO

~~(7) Proof. Let $J$ be well-ordered and let $J_0 \subseteq J$ be inductive. We want to show that $J_0 = J$.~~

(7) TODO ???

(8) (a) Proof. Let $A_1, A_2$ be well-ordered and define $<$ over $A_1 \cup A_2$ as $a < b$ if $a, b \in A_1$ and $a <_1 b$ or $a, b \in A_2$ and $a <_2 b$ or $a \in A_1$ and $b \in A_2$. Let $A \subseteq A_1 \cup A_2$ be an arbitrary subset. Set $l = \min A \setminus A_2$ (this is well-defined as $A_1$ is well-ordered). Let $b \in A$. If $b \in A_1$, then $l < b$ by construction. ~~If $b \in A_1$~~ ~~If $b \in A_2$, then~~ If ~~$b \notin A_1$ and~~ $b \in A_2$, then $l < b$, too. (As $l \in A_1$ by construction.) Thus, $A$ is well-ordered ~~und~~ under $<$.

$\square$

(b) Let $J$ be a well-ordered index set and let $\{A_j\}_{j \in J}$ be a family of ~~dis~~ disjoint well-ordered sets. We define $<$ over $A = \bigcup_{j \in J} A_j$ as follows:

$a < b$ if $a, b \in A_j$ for some $j \in J$;

or ~~$a, b$~~ $a \in A_j$, $b \in A_{j'}$ for $j, j' \in J$, $j < j'$

Then $A$ is well-ordered under $<$.

(9) (a) Proof. ~~Con~~ Let $n \in \mathbb{N}$ and consider

$$x = (\underbrace{2, 2, \ldots, 2}_{n+1 \text{ times}}, 1, 1, \ldots).$$

Then all elements of $A_x$ have the form

$$y = (y_1, y_2, \ldots, y_n, 1, 1, \ldots)$$

as $y_i = 1 = x_i$ for $i > n+1$ and ~~$y_i = 1 < 2 = x_i$~~ ~~$y_n = 1 < 2 =$~~ $y_{n+1} = 1 < 2 = x_{n+1}$. We can now define a bijective $f : (\mathbb{Z}_+)^n \to A_x$ as follows:

$$f(y) = (y_n, y_{n-1}, \ldots, y_1)$$

Clearly $f$ is order-preserving.

$\square$

(9) (b) Proof. Let $A_0 \subseteq A$ and pick some $a_0 \in A_0$. Then there is an $n \in \mathbb{N}$ such that $(a_0)_i = 1$ for all $i > n$. From (a), there is ~~some~~ some section of $A$, say $\tilde{A}$, that has the same order type as $\mathbb{Z}_+^2$. We have $a_0 \in \tilde{A}$ and as $\tilde{A}$ is well-ordered, $\tilde{A} \cap A_0$ has a minimum, say $l$. Clearly, $l$ is ~~at~~ also a minimum of $A_0$ and thus $A$ is well-ordered.

$\square$

(10) Theorem: Let $J$ and $C$ be well-ordered such that there is no surjection from a section of $J$ onto $C$. Then there is a <u>unique</u> $h: J \to C$ such that

$$h(x) = \min(C \setminus h(S_x)), \quad x \in J, \qquad (*)$$

where $S_x$ is the section of $J$ by $x$.

~~Proof.~~

~~(a) Let $h, k: D \to C$ be mappings with $D = S_x$ or $D = J$ for some fixed $x \in J$ satisfying $(*)$. Let $x, x' \in D$. Then~~

$$\cancel{h(x) = \min(C \setminus h(S_x))} \quad \cancel{\text{and}}$$

$$\cancel{k(\tilde{x}) = \min(C \setminus k(S_{\tilde{x}}))}.$$

~~Suppose $h(x) \neq k(\tilde{x})$ and w.l.o.g. $h(x) < k(x)$~~

~~(a) Let $h, k: D \to C$ be functions satisfying $(*)$ where $D = J$ or $D = S_y$ for some $y \in J$. Let $x \in D$ and suppose, for contradiction, $h(x) \neq k(x)$. W.l.o.g., let $h(x) < k(x)$. Then $h(x) \notin C \setminus k(S_x)$, so $h(x) \in k(S_x)$ as $h(x) \in C$. Similarly, $k(x) \notin k(S_x)$.~~

TODO

(11) TODO

11 The Maximum Principle

Theorem (Maximum Principle): Let $A$ be a set with a strict partial order $<$. Then there exists a maximal simply ordered subset $B \subseteq A$. That is, if $C \subseteq A$ is simply ordered, $B \subseteq C$.

Zorn's Lemma: Let $A$ be a strictly partially ordered set. If every simply ordered $B \subseteq A$ has an upper bound in $A$, then $A$ has a maximal element.

Exercises:

(1) Proof.

(i) Let $a \in \mathbb{R}$, then $a - a = 0$ which is not positive so $a \nless a$.

(ii) Let $a, b, c \in \mathbb{R}$ with $a < b$ and $b < c$. That is, $b - a \in \mathbb{Q}_+$ and $c - b \in \mathbb{Q}_+$. We thus have

$$c - a = \underbrace{c - b}_{\in \mathbb{Q}_+} + \underbrace{b - a}_{\in \mathbb{Q}_+} \in \mathbb{Q}_+,$$

so $a < b$.

□

The maximal simply ordered sets are

$$\{\alpha + x \mid x \in \mathbb{Q}_+\} = A_\alpha, \quad \alpha \in \mathbb{R} \setminus \mathbb{Q}.$$

(2) (a) Proof.

(i) Let $a \in A$. Then $a = a$, so $a \leq a$.

(ii) Let $a, b \in A$ such that ~~$b \leq a$. If~~ $a \leq b$ and $b \leq a$. Clearly neither $a < b$ nor $b < a$, so $a = b$.

(iii) Let $a, b, c \in A$ with $a \leq b$ and $b \leq c$. If $a = b$ or $b = c$, $a \leq c$ is clear. Otherwise, $a < c$ follows from transitivity of $<$, so $a \leq c$.

□

(b) Proof.

(i) Trivial.

(ii) Let $a, b, c \in A$ with $aSb$ and $bSc$. We essentially need to show that $a \neq c$. Suppose $a = c$. Then ~~$aSb$ and $bSa$, so~~ $aPb$ and $bPa$, so $a = b$. ↯ Thus, $a \neq c$ and we have $aSc$.

□

(3) It won't work for Example 1: Consider

$$\mathcal{A} = \{\{\}, \{0\}, \{1\}, \{0, 1\}\}$$

and pick $x = \{0, 1\}$. Then $\mathcal{B} = \mathcal{A}$, but $\{0\}$ and $\{1\}$ are not comparable.

It will ~~not~~ work for Example 2: Pick some point ~~$(x_0,$~~ $(x, y) \in \mathbb{R}^2$, then for two $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ both comparable to $(x_0, y_0)$, i.e., $x_0 = x_1$ and $x_0 = x_2$, are also comparable as $x_1 = x_2$.

(4) Let, for all $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$, $<$ be defined as

$$(x_0, y_0) < (x_1, y_1) \text{ iff } x_0 < x_1 \text{ and } y_0 \leq y_1.$$

The maximal ~~sum~~ simply ordered subsets are given by the sets

$$\{(x, y) \mid y = f(x)\}$$

where $f: \mathbb{R} \to \mathbb{R}$ is any monotonically increasing function.

(5) Proof. Let $\mathcal{A}$ be a collection of sets ~~and suppose that for~~ strictly partially ordered by proper inclusion $\subset$. Let Suppose that for every simply ordered $\mathcal{B} \subseteq \mathcal{A}$, $\bigcup_{B \in \mathcal{B}} B \in \mathcal{A}$. That is, ~~every~~ $\mathcal{B}$ is upper-bounded by an element in $\mathcal{A}$. Due to Zorn's Lemma, there is a maximal element $A \in \mathcal{A}$. In terms of $\subset$, there is no $B \in \mathcal{A}$ such that $A \subset B$. $\square$

(6) Proof. Let $\mathcal{A}$ be a collection of subsets of $X$ that is of finite type. Let $\mathcal{B} \subseteq \mathcal{A}$ be ~~properly~~ simply ordered (by proper inclusion). Set

$$C = \bigcup_{B \in \mathcal{B}} B.$$

As $\mathcal{A}$ is of finite type, we ~~have~~ have (for all $B \in \mathcal{B}$) that every finite $B_0 \subseteq B$ is in $\mathcal{A}$, i.e., $B_0 \in \mathcal{A}$. We therefore have

$$C = \bigcup_{B \in \mathcal{B}} \bigcup_{B_0 \subseteq B \text{ finite}} B_0$$

and thus ~~$\mathcal{C}$~~ $C \in \mathcal{A}$. Hence, by Kuratowski's Lemma, $\mathcal{A}$ has an element not properly contained in any other element of $\mathcal{A}$. $\square$

(7) Proof. Let $A$ be a set with partial strict ordering $<$. Let $\mathcal{A}$ be the collection of subsets of $A$ that are simply ~~and~~ ordered. That is, for all $B \in \mathcal{A}$ we have either $a<b$ or $b<a$ for all $a, b \in B$. ~~We want~~ We claim that $\mathcal{A}$ is of finite type. Then there is some $B \in \mathcal{A}$ such that $C \subset B$ for no $C \in \mathcal{A}$. Thus, $B$ is a maximal simply ordered subset of $A$.

We need to show that $\mathcal{A}$ is of finite ~~ty~~ type. Let $B \subseteq A$ be in $\mathcal{A}$, i.e., $B$ is simply ordered. Clearly every (finite) subset $B_0 \subseteq B$ is also ~~so~~ simply ordered, so $B_0 \in \mathcal{A}$. Conversely, let $B \subseteq A$ ~~such~~ such that for all ~~fin~~ finite $B_0 \subseteq B$, $B_0 \in \mathcal{A}$. Now let $a, b \in B$. Then $\{a, b\} \in \mathcal{A}$, so $a$ and $b$ are comparable. Thus, $B \in \mathcal{A}$ and $\mathcal{A}$ is of finite type.

□

(8) Note that we have

$$\begin{tikzcd}
\text{Maximum Principle} \arrow[r, Rightarrow] & \text{Zorn's Lemma} \arrow[d, Rightarrow] \\
\text{Tukey's Lemma} \arrow[u, Rightarrow] & \text{Kuratowski's Lemma} \arrow[l, Rightarrow]
\end{tikzcd}$$

So all the statements are equivalent!

Lemma (Kuratowski's): Let $\mathcal{A}$ be a collection of sets and suppose that for every $\mathfrak{B} \subseteq \mathcal{A}$ that is simply ordered by proper inclusion, $\bigcup_{B \in \mathfrak{B}} B \in \mathcal{A}$. Then there is some $B \in \mathcal{A}$ such that for no $C \in \mathcal{A}$, $C \subset B$.

Lemma (Tukey's): Let $\mathcal{A}$ be a collection of finite type, then there is some $B \in \mathcal{A}$ such that $C \subset B$ holds for no $C \in \mathcal{A}$.

Definition (Finite Type): A collection $\mathcal{A}$ of subsets of $X$ is of finite type if: a subset $B \subseteq X$ is in $\mathcal{A}$ if and only if every finite subset $B_0 \subseteq D$ is in $\mathcal{A}$.

(8) (a) *Proof.* Let $V$ be a vector space and let $A \subseteq V$ be linearly independent and let $v \in V$ such that $v \notin \operatorname{span} A$. Suppose $A \cup \{v\}$ is linearly dependent. Then there are coefficients $\alpha_1, \dots, \alpha_K, \alpha_v$ such that

$$\alpha_1 v_1 + \dots + \alpha_K v_K + \alpha_v v = 0$$

for suitable $v_1, \dots, v_K \in A$. Note that $\alpha_v \neq 0$ as ~~otherwise A was~~ $A$ is linearly independent. But then

$$v = -\frac{1}{\alpha_v}(\alpha_1 v_1 + \dots + \alpha_K v_K),$$

so $v \in \operatorname{span} A$. ↯

$\square$

(b) *Proof.* Let $V$ be a vector space and let $\mathcal{A}$ be the collection of all independent subsets of $V$. We show that $\mathcal{A}$ is of finite type. ~~Let $B \in \mathcal{A}$. That is, $B$ is linearly independent means~~ Let $B \subseteq V$. Suppose that $B \in \mathcal{A}$. Thus $B$ is linearly independent, i.e., every finite subset $B_0 \subseteq B$ is linearly independent. Hence, $B_0 \in \mathcal{A}$. Conversely, suppose that $B_0 \in \mathcal{A}$ for any finite $B_0 \subseteq B$. But then, by definition, $B$ is linearly independent, i.e., $B \in \mathcal{A}$. Hence, $\mathcal{A}$ is of finite type and there is a $B \in \mathcal{A}$ such that it is not properly contained by any other element of $\mathcal{A}$. That is, it is a maximal element of $\mathcal{A}$.

$\square$

(c) *Proof.* Let $V$ be a vector space and let $\mathcal{A}$ be the collection of linearly independent subsets of $V$. Let let $B$ be its maximal ~~ele~~ element. Let $v \in V$ be arbitrary and suppose that $v \notin \operatorname{span} B$. But then $B \cup \{v\}$ would be linearly independent while properly including $B$. ↯ Thus, $v \in \operatorname{span} B$ and $\operatorname{span} B = V$, i.e., $B$ is a basis.

$\square$

Supplementary Exercises: SKIPPED