12 Topological Spaces

Definition (Topology): A <u>topology</u> on a set X is a collection $\mathcal{T}$ of subsets of X such that:

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$.

(ii) For all $\mathcal{A} \subseteq \mathcal{T}$, $\bigcup_{A \in \mathcal{A}} A \in \mathcal{T}$.

(iii) For all <u>finite</u> $\mathcal{A} \subseteq \mathcal{T}$, $\bigcap_{A \in \mathcal{A}} A \in \mathcal{T}$.

A set X for which a topology $\mathcal{T}$ is defined, or, more ~~pre~~ precisely the tuple $(X, \mathcal{T})$, is called a <u>topological space</u>.

Definition (Open Set): Let $(X, \mathcal{T})$ be a topological space. A set $U \subseteq X$ is <u>open</u> if $U \in \mathcal{T}$.

Definition (Discrete/Trivial Topology): Let X be a set. The topology $\mathcal{T}$ of <u>all</u> subsets of X is called the <u>discrete topology</u> and $\{\emptyset, X\}$ is called the <u>trivial topology</u>.

Definition (Finer/Coarser/Comparable): Let X be a set and let $\mathcal{T}, \mathcal{T}'$ be two topologies over X. If $\mathcal{T}' \supseteq \mathcal{T}$, we say that $\mathcal{T}'$ is <u>finer</u> than $\mathcal{T}$. If $\mathcal{T}' \supset \mathcal{T}$, we say $\mathcal{T}'$ is <u>strictly</u> finer than $\mathcal{T}$. We also say that $\mathcal{T}$ is (strictly) <u>coarser</u> than $\mathcal{T}'$. We say $\mathcal{T}$ and $\mathcal{T}'$ are <u>comparable</u> if $\mathcal{T}' \supseteq \mathcal{T}$ or $\mathcal{T} \supseteq \mathcal{T}'$.

13 Basis for a Topology

Definition (Basis): Let X be a set, then a collection $\mathcal{B}$ of subsets of X (called <u>basis elements</u>) is a basis if:

(i) For all $x \in X$, there is at least one $B \in \mathcal{B}$ such that $x \in B$.

(ii) If $x \in B_1 \cap B_2$ for some $B_1, B \in \mathcal{B}$, there is a $B_3 \in \mathcal{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

We define a <u>topology generated by $\mathcal{B}$</u> as follows: For all $U \subseteq X$ we have $U \in \mathcal{T}$ if for all $x \in U$ there is a $B \in \mathcal{B}$ such that $x \in B$ and $U \subseteq B$. In particular, $\mathcal{T} \supseteq \mathcal{B}$.

Lemma: Let X be a set, let $\mathcal{B}$ be a basis and let $\mathcal{T}$ be the generated topology. Then

$$\mathcal{T} = \{ \textstyle\bigcup_{B \in \mathcal{B}'} B \mid \mathcal{B}' \subseteq \mathcal{B} \},$$

i.e., $\mathcal{T}$ is the collection of all unions of elements in $\mathcal{B}$.

Lemma (Basis from Topology): let $(X, \mathcal{T})$ be a topological space and let $\mathcal{C}$ be a collection of open sets such that for each open set $U \subseteq X$ and each $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C$ and $C \subseteq U$. Then $\mathcal{C}$ is a basis of $\mathcal{T}$.

Lemma (Finer by Basis): let X be ~~as~~ a set and let $\mathcal{B}$ and $\mathcal{B}'$ be basis for topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for each $x \in X$ and each $B \in \mathcal{B}$ with $x \in B$ there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.

Definition/Lemma (Topologies on $\mathbb{R}$): let $\mathcal{B}$ be the collection of all open intervals in the real line, (a, b), then the topology generated by $\mathcal{B}$ is the <u>standard topology</u> on $\mathbb{R}$. If $\mathcal{B}'$ is the collection of all half-open intervals $[a, b)$, then the topology generated by $\mathcal{B}'$ is called the <u>lower limit topology</u> on $\mathbb{R}$, denoted by $\mathbb{R}_\ell$. Let $K = \{1/n \mid n \in \mathbb{Z}_+\}$. If $\mathcal{B}''$ is the collection of all open intervals (a, b) along with all sets of the form $(a, b) \setminus K$, then the topology generated by $\mathcal{B}''$ is called the <u>K-topology</u> on $\mathbb{R}$, denoted by $\mathbb{R}_K$.

The topologies $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are strictly finer than $\mathbb{R}$, but are not comparable to each other.

~~Definition (Subbasis): let X (X, T) be a topological space.~~

~~Definition (Subbasis): let X be a set and let S be a collection of subsets of X whose union equals X. Then S is a subbasis and the topology generated by S is the collection of all unions of finite intersections of S of elements of S.~~

Definition (Subbasis): let X be a set, then a collection $\mathcal{S}$ of subsets of X is a <u>subbasis</u> if:

(i) For all $x \in X$, there is at least one $S \in \mathcal{S}$ such that $x \in S$.

We define the <u>topology generated by $\mathcal{S}$</u> as the collection of ~~all~~ all unions of finite intersections of elements of $\mathcal{S}$.

Exercises:

(1) Proof. Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Suppose that for all $x \in A$ there is an $U \in \mathcal{T}$ with $x \in U$ such that $U \subseteq A$. ~~Denote for all $x \in A$ a set~~ Let, for all $x \in A$, be $U_x \in \mathcal{T}$ ~~the~~ a set such that $x \in U_x \subseteq A$. We claim that

$$A = \bigcup_{x \in A} U_x.$$

"$\subseteq$": Let $y \in A$. Then $y \in U_y$, so $y \in \bigcup_{x \in A} U_x$. "$\supseteq$": As for all $x \in A$, $U_x \subseteq A$, we have $A \supseteq \bigcup_{x \in A} U_x$. Thus A is the union of open sets, so $A \in \mathcal{T}$. $\square$

(2) We have the following topologies:

$\mathcal{T}_{11} = \{\emptyset, \{a, b, c\}\}$

$\mathcal{T}_{12} = \{\emptyset, \{a, b, c\}, \{a\}, \{a, b\}\}$

$\mathcal{T}_{13} = \{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}\}$

$\mathcal{T}_{21} = \{\emptyset, \{a, b, c\}, \{b\}\}$

$\mathcal{T}_{22} = \{\emptyset, \{a, b, c\}, \{a\}, \{b, c\}\}$

$\mathcal{T}_{23} = \{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{b\}, \{c\}\}$

$\mathcal{T}_{31} = \{\emptyset, \{a, b, c\}, \{a, b\}\}$

$\mathcal{T}_{32} = \{\emptyset, \{a, b, c\}, \{a, b\}, \{a\}, \{b\}\}$

$\mathcal{T}_{33} = \{\emptyset, \{a, b, c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a\}, \{b\}, \{c\}\}$

	$\mathcal{T}_{11}$	$\mathcal{T}_{12}$	$\mathcal{T}_{13}$	$\mathcal{T}_{21}$	$\mathcal{T}_{22}$	$\mathcal{T}_{23}$	$\mathcal{T}_{31}$	$\mathcal{T}_{32}$	$\mathcal{T}_{33}$
$\mathcal{T}_{11}$	e	c	c	c	c	c	c	c	c
$\mathcal{T}_{12}$	f	e	-	-	-	-	f	c	c
$\mathcal{T}_{13}$	f	-	e	f	-	c	f	-	c
$\mathcal{T}_{21}$	f	-	c	e	-	c	-	c	c
$\mathcal{T}_{22}$	f	-	-	-	e	-	-	-	c
$\mathcal{T}_{23}$	f	-	f	f	-	e	f	-	c
$\mathcal{T}_{31}$	f	c	c	-	-	c	e	c	c
$\mathcal{T}_{32}$	f	f	-	f	-	-	f	e	c
$\mathcal{T}_{33}$	f	f	f	f	f	f	f	f	e

Read: "row is e/c/f/- than column"

- e equal
- c coarser
- f finer
- \- not comparable

(3) Proof. Let X be a set and let

$$\mathcal{T}_c = \{ U \subseteq X \mid X \setminus U \text{ is countable or } X \}.$$

We check that $\mathcal{T}_c$ is a topology.

(i) As $\cancel{\emptyset}$ $X \setminus \emptyset = X$, $\emptyset \in \mathcal{T}_c$.
As $X \setminus X = \emptyset$ and $\emptyset$ is finite, $X \in \mathcal{T}_c$.

(ii) Let $\mathcal{A} \subseteq \mathcal{T}_c$, then

$$\cancel{\bigcup_{A \in \mathcal{A}} A}$$

$$X \setminus \bigcup_{A \in \mathcal{A}} A = \bigcap_{A \in \mathcal{A}} (X \setminus A).$$

This is the intersection of only countable sets (or X), so the result is countable (or X). Thus, $\bigcup_{A \in \mathcal{A}} A \in \mathcal{T}_c$.

(iii) Let $A_1, \dots, A_k \in \mathcal{T}_c$. Then

$$\cancel{X \setminus A}$$

$$X \setminus (A_1 \cap \cdots \cap A_k) = (X \setminus A_1) \cup \cdots \cup (X \setminus A_k).$$

This is a finite union of countable sets (or X), ~~c~~ so the result is countable (or X). ~~Thus~~ Thus, $A_1 \cap \cdots \cap A_k \in \mathcal{T}_c$.

Hence, $\mathcal{T}_c$ is a topology on X.

$\square$

No, the collection

$$\mathcal{T}_\infty = \{ U \subseteq X \mid X \setminus U \text{ is infinite or empty or } X \}$$

is not a topology. Consider $X = \mathbb{Z}_+$. Then we have the sets $A_1 = \{2, 4, 6, \dots\}$ and $A_2 \in \{3, 5, 7, \dots\}$ ~~both~~ both in $\mathcal{T}_\infty$. That is, the sets of even/odd positive integers without one. However, $A_1 \cup A_2 = \{2, 3, 4, \dots\}$ is not in $\mathcal{T}_\infty$ as $\mathbb{Z}_+ \setminus (A_1 \cup A_2) = \{1\}$ is finite and not empty or all of $\mathbb{Z}_+$.

(4) (a) Proof. Let X be a set and let $\{\mathcal{T}_\alpha\}$ be a collection of topologies ~~on~~ on X. We want to show that $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ is a topology.

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$ are clear.

(ii) Let $\mathcal{A} \subseteq \mathcal{T}$. Then, for all $\mathcal{T}_\alpha$, we have

$$\bigcup_{A \in \mathcal{A}} A \in \mathcal{T}_\alpha.$$

Thus, also $\bigcup_{A \in \mathcal{A}} A \in \mathcal{T}$.

(iii) Let $A_1, \dots, A_k \in \mathcal{T}$. Then, for all $\mathcal{T}_\alpha$, we have

$$A_1 \cap \dots \cap A_k \in \mathcal{T}_\alpha.$$

Thus, also $A_1 \cap \dots \cap A_k \in \mathcal{T}$.

Hence, $\mathcal{T}$ is a topology on X. $\square$

~~No, $\bigcup \mathcal{T}_\alpha$ is not a topology. Consider $X = \{a, b\}$ and~~

~~$\mathcal{T}_1 = \{\emptyset, \{a, b\}, \{a\}\}$,~~

~~$\mathcal{T}_2 = \{\emptyset, \{a, b$~~

No, $\bigcup \mathcal{T}_\alpha$ is not a topology. Consider $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, \{a, b, c\}, \{a\}\},$$

$$\mathcal{T}_2 = \{\emptyset, \{a, b, c\}, \{b\}\}.$$

These are clearly topologies ~~on~~ on X, but $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology as $\{a\}, \{b\} \in \mathcal{T}_1 \cup \mathcal{T}_2$ but $\{a, b\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$.

~~(4) (b) Proof. Let X be a set and let $\{\mathcal{T}_\alpha\}$ be a collection of topologies on X. Then there is a smallest topology on X such that~~ Claim.

(4) (b) Claim. Let X be a set and let $\{\mathcal{T}_\alpha\}$ be a collection of topologies on X. Then:

(i) ~~Then~~ There is a smallest topology $\mathcal{T}$ on X such that for all $\mathcal{T}_\alpha$, $\mathcal{T} \supseteq \mathcal{T}_\alpha$. It is smallest in that for all $\mathcal{T}'$ with said property, $\mathcal{T} \subseteq \mathcal{T}'$.

(ii) There is a largest topology $\mathcal{T}$ on X such that for all $\mathcal{T}_\alpha$, $\mathcal{T}_\alpha \supseteq \mathcal{T}$. It is largest in that for all $\mathcal{T}'$ with said property, ~~$\mathcal{T}' \subseteq$~~ $\mathcal{T} \supseteq \mathcal{T}'$.

Both smallest and largest topology are unique.

Proof. For both cases, uniqueness follows directly from the smallest/largest property. We continue by showing that there are such topologies.

(ii) Set $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. We showed in (a) that $\mathcal{T}$ is a topology. Clearly, $\mathcal{T}_\alpha \supseteq \mathcal{T}$. Let $\mathcal{T}'$ be another topology such that $\mathcal{T}_\alpha \supseteq \mathcal{T}'$. We want to show $\mathcal{T} \subseteq \mathcal{T}'$. Let $A \in \mathcal{T}$. Then, by construction, $A \in \mathcal{T}_\alpha$ for all $\mathcal{T}_\alpha$. As $\mathcal{T}' \subseteq \mathcal{T}_\alpha$ for all $\mathcal{T}_\alpha$, also $A \in \mathcal{T}'$. Hence, $\mathcal{T} \subseteq \mathcal{T}'$.

~~(i) Set $\tilde{\mathcal{S}} = \bigcup \mathcal{T}_\alpha$ and treat $\tilde{\mathcal{S}}$ as a subbasis. (It is one~~

(i) Set $\mathcal{S} = \bigcup \mathcal{T}_\alpha$. Then $\mathcal{S}$ is a subbasis as for all $\mathcal{T}_\alpha$, $X \in \mathcal{T}_\alpha$, so $\bigcup_{S \in \mathcal{S}} S = X$. ~~Moreover,~~ Denote by $\mathcal{T}$ the topology generated by $\mathcal{S}$. Clearly, as $\mathcal{S} \subseteq \mathcal{T}$, we have $\mathcal{T} \supseteq \mathcal{T}_\alpha$ for all $\mathcal{T}_\alpha$. Now suppose $\mathcal{T}'$ is another topology with $\mathcal{T}' \supseteq \mathcal{T}_\alpha$ for all $\mathcal{T}_\alpha$. We show that $\mathcal{T} \subseteq \mathcal{T}'$. Let $A \in \mathcal{T}$. ~~Then there are~~ ~~Then there is a family $\mathcal{B} \subseteq \mathcal{S}$ such that $A = \bigcup_{B \in \mathcal{B}} B$.~~ ~~Then there are elements $S^1_{\beta}, \ldots$~~ ~~$S^1_\beta, S^2_\beta, \ldots, S^{k_\beta}_\beta \in \mathcal{S}$~~ Then there is a family $\mathcal{B}$ such that $A = \bigcup_{B \in \mathcal{B}} B$ where for each $B \in \mathcal{B}$ there are elements $S^B_1, S^B_2, \ldots, S^B_{k_B} \in \mathcal{S}$ such that $B = S^B_1 \cap \cdots \cap S^B_{k_B}$. But then also $S^B_1, \ldots, S^B_{k_B} \in \mathcal{T}'$ as $\mathcal{T}' \supseteq \mathcal{T}_\alpha$ for all $\mathcal{T}_\alpha$ and $\mathcal{S} = \bigcup \mathcal{T}_\alpha$. Thus, as $\mathcal{T}'$ is a topology, ~~$B \in \mathcal{T}'$ for a~~ $\mathcal{B} \subseteq \mathcal{T}'$, so $A \in \mathcal{T}'$. Hence, we have $\mathcal{T} \subseteq \mathcal{T}'$.

$\square$

(4) (c) Consider $X = \{a, b, c\}$ and

$$\mathcal{T}_1 = \{\emptyset, X, \{a\}, \{a, b\}\},$$

$$\mathcal{T}_2 = \{\emptyset, X, \{a\}, \{b, c\}\}.$$

Then the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_\ell = \{\emptyset, X, \{a\}\}.$$

The smallest topology ~~cont~~ containing $\mathcal{T}_1$ and $\mathcal{T}_2$ is

$$\mathcal{T}_s = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}.$$

~~(5) Proof. Let X be a set and let $\mathcal{A}$ be a basis. Denote by T the set collection of of all topologies contain containing $\mathcal{A}$, i.e.,~~

$$\cancel{\{ \ T = \{\mathcal{T} \subseteq 2^X \mid \mathcal{T} \text{ topology}, \mathcal{A} \subseteq \mathcal{T}\}.}$$

~~Denote by $\mathcal{T}$ the to $\mathcal{T}(\mathcal{A})$ the topology generated by $\mathcal{A}$. We show $\mathcal{T}(\mathcal{A}) = \bigcap_{\mathcal{T} \in T} \mathcal{T}$. "$\subseteq$": As As As as as As As as $\mathcal{A} \subseteq \mathcal{T}(\mathcal{A})$, $\mathcal{T}(\mathcal{A}) \in T$, so $\mathcal{T}(\mathcal{A}) \in \subseteq \bigcup_{\mathcal{T} \in T} \mathcal{T}$ is clear. "$\supseteq$": Let $A \in \bigcup_{\mathcal{T} \in T} \mathcal{T}$. As $A \subseteq X$, for each $x \in A$ there is an $B_x \in \mathcal{A}$ with $x \in B_x$. As $\mathcal{T}(\mathcal{A})$ is the union collection of all unions of elements of $\mathcal{A}$, we have~~

$$\cancel{\bigcup_{x \in A} B_x \in \mathcal{T}(\mathcal{A}),}$$

~~so $\bigcup_{\mathcal{T} \in T} \mathcal{T} \subseteq \mathcal{T}(\mathcal{A})$.~~

□

~~(6) Proof. Let X be a set and let $\mathcal{A}$ be a subbasis. De Denote by T and $\mathcal{T}(\mathcal{A})$ the set of topologies containing $\mathcal{A}$ and the topology generated by $\mathcal{A}$, respectively. We show that $\mathcal{T}(\mathcal{A}) =$~~

(5) Proof. Let X be a set, let $\mathcal{A}$ be a basis, let $\mathcal{T}(\mathcal{A})$ be the topology generated by $\mathcal{A}$, and let T be the collection of all topologies ~~generated~~ containing $\mathcal{A}$. We show that $\mathcal{T}(\mathcal{A}) = \bigcap_{\mathcal{T}\in T}\mathcal{T}$. "$\supseteq$" is clear as $\mathcal{T}(\mathcal{A})\in T$. "$\subseteq$": Let $A\in\mathcal{T}(\mathcal{A})$. Then there is a family $\{B_\alpha\}\subseteq\mathcal{A}$ such that $A=\bigcup B_\alpha$. As $\mathcal{A}\subseteq\mathcal{T}$ for all $\mathcal{T}\in T$, also $B_\alpha\in\mathcal{T}$ for all B_α and all $\mathcal{T}\in T$. Thus, $\{B_\alpha\}\subseteq\bigcap_{\mathcal{T}\in T}\mathcal{T}$ and as the intersection of topologies is a topology, $A=\bigcup B_\alpha\in\bigcap_{\mathcal{T}\in T}\mathcal{T}$.

$\square$

Proof. Let X be a set, let $\mathcal{A}$ be a subbasis, let $\mathcal{T}(\mathcal{A})$ be the topology generated by $\mathcal{A}$, and let T be the collection of all topologies containing $\mathcal{A}$. We show that $\mathcal{T}(\mathcal{A}) = \bigcap_{\mathcal{T}\in T}\mathcal{T}$. "$\supseteq$" is clear as $\mathcal{T}(\mathcal{A})\in T$. "$\subseteq$": Let $A\in\mathcal{T}(\mathcal{A})$. Denote by $\mathcal{B}$ the ~~set of~~ family of all finite intersections of elements of $\mathcal{A}$. Then there is a collection $\{B_\alpha\}\subseteq\mathcal{B}$ such that $A=\bigcup B_\alpha$. As $\mathcal{A}\subseteq\mathcal{T}$ for all $\mathcal{T}\in T$ and topologies are closed under finite intersections, also $\mathcal{B}\subseteq\mathcal{T}$ for all $\mathcal{T}\in T$. Thus, $B_\alpha\in\mathcal{T}$ for all B_α and all $\mathcal{T}\in T$. Thus, $\{B_\alpha\}\subseteq\bigcap_{\mathcal{T}\in T}\mathcal{T}$ and as the intersection of topologies is a topology, $A=\bigcup B_\alpha\in\bigcap_{\mathcal{T}\in T}\mathcal{T}$.

$\square$

(6) Proof. To show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable, we show that there are subsets $A_\ell, A_K\subseteq\mathbb{R}$ that are open w.r.t. $\mathcal{T}_\ell$ and $\mathcal{T}_K$ but not w.r.t. $\mathcal{T}_K$ and $\mathcal{T}_\ell$, respectively, where $\mathcal{T}_\ell$ and $\mathcal{T}_K$ denote the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$, respectively. ~~We begin by showing that that not~~ $\cancel{\mathcal{T}_K\subseteq\mathcal{T}_\ell}$. $\cancel{\mathcal{T}_K\supseteq\mathcal{T}_\ell}$. ~~Let~~ Let $\mathcal{B}_\ell$ and $\mathcal{B}_K$ be the respective bases.

We begin by showing that <u>not</u> $\mathcal{T}_\ell\supseteq\mathcal{T}_K$. Consider the basis element $(-1,1)\setminus K\in\mathcal{B}_K$ and $x=0\in\mathbb{R}$. Clearly there is no interval $[a,b)\ni 0$ such that ~~it is so~~ $[a,b)\in(-1,1)\setminus K$. (We have $b>0$ so $1/n\in[a,b)$ for some $n\in\mathbb{Z}_+$, but $1/n\notin(-1,1)\setminus K$.) Thus, $\mathcal{T}_\ell$ is not finer than $\mathcal{T}_K$.

We now show that <u>not</u> $\mathcal{T}_K\supseteq\mathcal{T}_\ell$. Consider $0\in\mathbb{R}$ and the basis element $[0,1)\in\mathcal{B}_\ell$. Clearly there is no open interval (a,b) with $0\in(a,b)\subseteq[0,1)$. Similarly there is no $\cancel{(a,b)\setminus K}$ $0\in(a,b)\setminus K\subseteq[0,1)$. Thus, $\mathcal{T}_K$ is not finer than $\mathcal{T}_\ell$.

$\square$

(7)

	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$
$\mathcal{T}_1$	=	c	-	c	c
$\mathcal{T}_2$	f	=	-	-	f
$\mathcal{T}_3$	-	-	=	-	-
$\mathcal{T}_4$	f	-	-	=	-
$\mathcal{T}_5$	f	c	-	-	=

Read: "row is =/c/f/- to column"

= equal
c coarser
f finer
- not comparable

(8) (a) *Proof.* Consider $\mathcal{B} = \{(a,b) \mid a<b,\ a,b \in \mathbb{Q}\}$. We show that $\mathcal{T}(\mathcal{B})$ is the standard topology on $\mathbb{R}$. Let $\mathcal{T}$ be the standard topology and let $U \in \mathcal{T}$. Let $x \in U$. As U is the union of open sets, there are $a, b \in \mathbb{R}$, $a<b$ such that $x \in (a,b) \subseteq U$. As $\mathbb{Q}$ is dense in $\mathbb{R}$, we can choose $\tilde{a}, \tilde{b} \in \mathbb{Q}$ with

$$a < \tilde{a} < x < \tilde{b} < b.$$

Thus $(\tilde{a}, \tilde{b}) \subseteq (a,b) \subseteq U$ where $(\tilde{a}, \tilde{b}) \in \mathcal{B}$. Due to Lemma 13.2, $\mathcal{B}$ is a basis of $\mathcal{T}$ so $\mathcal{T}(\mathcal{B}) = \mathcal{T}$. $\square$

~~(b) Proof. Consider $\mathcal{C} = \{[a,b) \mid a<b,\ a,b \in \mathbb{Q}\}$. Let $(\mathbb{R}, \mathcal{T}_\ell)$ be the lower limit topological space. Let $U \in \mathcal{T}_\ell$ and $x \in U$. Let $a,b \in \mathbb{R}$, $a<b$, a irrational. Then $(a,b) \in \mathcal{T}$ Let $a,b \in \mathbb{R}$, $a<b$, a irrational. Then $[a,b) \in \mathcal{T}_\ell$.~~

(b) *Proof.* Consider $\mathcal{C} = \{[a,b) \mid a<b,\ a,b \in \mathbb{Q}\}$. We first show that $\mathcal{C}$ is a basis for a topology on $\mathbb{R}$. Let $x \in \mathbb{R}$, then there are $a,b \in \mathbb{Q}$ such that ~~$x-1<$~~ $a<x<b$ ~~$<x+1$~~ as $\mathbb{Q}$ is dense in $\mathbb{R}$. Then $x \in [a,b) \in \mathcal{C}$. ~~Now let $B_1 = (a_1, b$ $B_1 = [a_1, b_1)$, $B_2 = [a_2, b_2) \in \mathcal{C}$. If $b_1 \leq a_2$, then $B_1 \cap B_2 = \emptyset$~~ Let $x \in \mathbb{R}$ and let B_1 ~~$B_1 = (a_1, b_1$~~ $B_1 = [a_1, b_1)$, $B_2 = [a_2, b_2) \in \mathcal{C}$ such that $x \in B_1$ and $x \in B_2$. Then* ~~$B_1 \cap B_2 = [a_2,$~~ $B_1 \cap B_2 = [a_2, b_1) \in \mathcal{C}$, so $\mathcal{C}$ is a basis for a topology $\mathcal{T}(\mathcal{C})$ on $\mathbb{R}$. We now show that $\mathcal{T}(\mathcal{C}) \neq \mathcal{T}_\ell$ where $\mathcal{T}_\ell$ is the lower limit topology on $\mathbb{R}$. Let $\mathcal{B}$ be the lower limit topology basis. Consider $[a,b) \in \mathcal{B}$ with a irrational. Then for all $[\tilde{a}, \tilde{b}) \in \mathcal{C}$ with $a \in [\tilde{a}, \tilde{b})$ we have $\tilde{a} < a$ as a is irrational while $\tilde{a}$ is rational. But then $[\tilde{a}, \tilde{b}) \nsubseteq [a,b)$, so <u>not</u> $\mathcal{T}(\mathcal{C}) \supseteq \mathcal{T}_\ell$. Thus, $\mathcal{T}(\mathcal{C}) \neq \mathcal{T}_\ell$, so $\mathcal{C}$ generates a different topology. $\square$

* Or $B_1 \cap B_2 = [a_1, b_2) \in \mathcal{C}$.

14 ## The Order Topology

Definition (Order Topology): Let X be a simply ordered set with ~~at least two~~ more than one element. Let $\mathcal{B}$ be the collection of all sets of the following types:

(i) all open intervals (a, b) in X;

(ii) all intervals $[a_0, b)$ where $a_0 \in X$ is the minimum of X (if any);

(iii) all intervals $(a, b_0]$, where $b_0 \in X$ is the maximum of X (if any).

Then $\mathcal{B}$ is the basis for the <u>order topology</u> on X.

15 ## The Product Topology

Definition (Product Topology): Let X and Y be topological spaces. The <u>product topology</u> on $X \times Y$ is the topology generated by

$$\mathcal{B} = \{ U \times V \subseteq X \times Y \mid U \text{ and } V \text{ open} \}.$$

Theorem (Product Basis): Let X and Y be topological spaces with bases $\mathcal{B}$ and $\mathcal{C}$, respectively. Then

$$\mathcal{D} = \{ B \times C \mid B \in \mathcal{B},\ C \in \mathcal{C} \}$$

is a basis for the topology on $X \times Y$.

Definition (Projection): Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be defined by the ~~equ~~ equations

$$\pi_1(x, y) = x \qquad \text{and} \qquad \pi_2(x, y) = y.$$

The maps π_1 and π_2 are called <u>projections</u>.

Theorem (Product Subbasis): Let X and Y be topological spaces. Then the collection

$$\mathcal{S} = \{ \pi_1^{-1}(U) \mid U \subseteq X \text{ open} \} \cup \{ \pi_2^{-1}(V) \mid V \subseteq Y \text{ open} \}$$

is a subbasis for the product topology on $X \times Y$.

16 The Subspace Topology

Definition (Subspace Topology): Let $(X, \mathcal{T})$ be a topological space and let $Y \subseteq X$. Then the collection

$$\mathcal{T}_Y = \{ Y \cap U \mid U \in \mathcal{T} \}$$

is the <u>subspace topology</u>. With this topology, Y is called a <u>subspace</u> of X.

Lemma (Subspace Basis): Let $\mathcal{B}$ be a basis for the topology on X and let $Y \subseteq X$. Then

$$\cancel{\mathcal{B}_Y = \{ B \cap Y} \quad \mathcal{B}_Y = \{ Y \cap B \mid B \in \mathcal{B} \}$$

is a basis for the subspace topology on Y.

Lemma (Open Sets in Subspace): Let Y be a subspace of X. If U is open in Y and Y is open in X, then U is open in X.

~~Theorem (Subspace & Product Topologies): Let A and B be subspaces of X and Y, respectively. Then the product topology on A×B is the same as the topology subspace topology induced by A×B disregarded as a subset of X×Y and the respective product topology.~~

Theorem (Subspace & Order Topologies): Let ~~X~~ X be an ordered set in the order topology. Let $Y \subseteq X$ be convex. Then the order topology on Y is the same as the topology Y inherits as a subspace of X.

Theorem (Subspace & Product Topologies): Let X and Y be topological spaces and let ~~x~~ $A \subseteq X$ and $B \subseteq Y$ be subspaces. Then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $X \times Y$.

Exercises:

(1) *Proof.* Let X be a topological space, let $Y \subseteq X$ be a subspace of ~~X~~ X and let $A \subseteq Y$. Denote by $\mathcal{T}_A$ and $\mathcal{T}_A'$ the topologies A inherits from X and Y, respectively. Denote by $\mathcal{T}$ and $\mathcal{T}_Y$ the topologies of X and Y, respectively. We show that $\mathcal{T}_A = \mathcal{T}_A'$. "$\subseteq$": Let $U \in \mathcal{T}_A$. Then there is ~~an~~ a $V \in \mathcal{T}$ such that $U = A \cap V$. But as $A \subseteq Y$, $A \cap V = A \cap Y \cap V$, where $Y \cap V \in \mathcal{T}_Y$ by construction. Thus, $U \in \mathcal{T}_A'$, too. "$\supseteq$": Let $U \in \mathcal{T}_A'$. Then there is a $V \in \mathcal{T}_Y$ such that $U = A \cap V$. There is also a $V' \in \mathcal{T}$ such that $V = Y \cap V'$. Hence, $U = A \cap V = A \cap Y \cap V' = A \cap V'$ as $A \subseteq Y$. Therefore, $U \in \mathcal{T}_A$, too. $\square$

(2) Claim. Let X be a ~~set~~ set with topologies $\mathcal{T}, \mathcal{T}'$, where $\mathcal{T}'$ is finer than $\mathcal{T}$. ~~&~~ Let $Y \subseteq X$. Then $\mathcal{T}'_Y$ is finer than $\mathcal{T}_Y$, where $\mathcal{T}'_Y$ and $\mathcal{T}_Y$ are the corresponding subspace topologies. (If $\mathcal{T}'$ is strictly finer than $\mathcal{T}$, $\mathcal{T}'_Y$ is not nec. $\mathcal{T}_Y$.)

Proof. Let $U \in \mathcal{T}_Y$, then there is a $V \in \mathcal{T}$ with $U = Y \cap V$. But as $\mathcal{T}' \supseteq \mathcal{T}$, also $V \in \mathcal{T}'$, so $U \in \mathcal{T}'_Y$, too. $\square$

(3) Due to Theorem 16.4, the order topology on $Y = [-1, 1]$ is the same as the topology Y inherits from $\mathbb{R}$. Thus,

A is open (and a basis element)

B is open (and a basis element)

C is not open (every basis element containing $1/2$ also contains a smaller number)

D is not open (same reason)

~~E is not open (every basis element containing 0 also contains~~

E is open

Proof. Consider $Y = [-1, 1]$ and the order topology $\mathcal{T}_Y$ generated by the respective basis. Consider

$$E = \{ x \mid 0 < |x| < 1,\ 1/x \notin \mathbb{Z}_+ \}.$$

We show that

$$E = (-1, 0) \cup \bigcup_{n \in \mathbb{Z}_+} (1/(n+1), 1/n).$$

"$\subseteq$": Let $x \in E$. If $x < 0$, clearly $x \in RHS$ as $x \in (-1, 0)$. ~~Sup~~ Suppose $x > 0$. Then there is no $n \in \mathbb{Z}_+$ with ~~$x =$~~ $1/x = n$, i.e., $x = 1/n$. Thus, by the Archimedean principle, there is an $m \in \mathbb{Z}_+$ such that ~~w~~ $m < 1/x < m+1$. That is, $1/(m+1) < x < 1/m$, so ~~$x\in$~~ $x \in RHS$. "$\supseteq$": Let $x \in RHS$. If $x < 0$, $x \in E$ is clear as ~~$1/x < 0$~~ $0 > 1/x \notin \mathbb{Z}_+$. Suppose $x > 0$. Then there is an $n \in \mathbb{Z}_+$ with $x \in (1/(n+1), n)$, so neither $x = 1/(n+1)$ ~~nor~~ nor $x = 1/n$. Hence, $x \in E$. We can thus construct E from ~~elem~~ basis elements so it is open. $\square$

(4) Proof. Let X and Y be topological spaces with bases $\mathcal{B}$ and $\mathcal{C}$, ~~as~~ respectively. Let $\mathcal{D}$ be the basis for the product topology on $X \times Y$. ~~Let $U \times V$~~ Let $U \times V \subseteq X \times Y$ be open. Then there are basis elements $\{B_\alpha\} \subseteq \mathcal{B}$ and $\{C_{\alpha}\} \subseteq \mathcal{C}$ such that

$$U \times V = \bigcup_\alpha B_\alpha \times C_\alpha .$$

But then

$$\pi_1(U \times V) = \pi_1\left(\bigcup_\alpha B_\alpha \times C_\alpha\right) = \bigcup_\alpha \pi_1(B_\alpha \times C_\alpha) = \bigcup_\alpha B_\alpha$$

which is a union over basis elements of X, so $\pi_1(U \times V)$ is ~~a~~ open, too. Thus, π_1 is an open map (and so is π_2). $\square$

~~(5) (a) Proof. Let $\mathcal{T}, \mathcal{T}', \mathcal{U}, \mathcal{U}'$ be topologies and let $X \in \mathcal{T}$, $X' \in \mathcal{T}'$, $Y \in \mathcal{U}$, $Y' \in \mathcal{U}'$ be all nonempty. Suppose that $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$.~~

(5) ~~TODO~~ → p. 18

(6) Proof. Consider $\mathbb{R}^2$ under the standard topology and define

$$\mathcal{C} = \{(a,b) \times (c,d) \mid a < b,\ \ c < d,\ \ a,b,c,d \in \mathbb{Q}\}.$$

Let $U \subseteq \mathbb{R}^2$ be open and let $x \times y \in U$. Then there are $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \in \mathbb{R}$ such that $x \times y \in (\tilde{a}, \tilde{b}) \times (\tilde{c}, \tilde{d}) \subseteq U$ as the open rectangles are a basis. As $\mathbb{Q}$ is dense in $\mathbb{R}$, there are now $a,b,c,d \in \mathbb{Q}$ such that

$$\tilde{a} < a < x < b < \tilde{b} \quad \text{and}$$

$$\tilde{c} < c < y < d < \tilde{d}.$$

Thus, $x \in (a,b) \times (c,d) \subseteq (\tilde{a}, \tilde{b}) \times (\tilde{c}, \tilde{d}) \subseteq U$, so $\mathcal{C}$ is a basis for the standard topology on $\mathbb{R}^2$. $\square$

(7) Wg. Consider $X = \mathbb{Q}$ and the set

$$A = \{ x \in X \mid x^2 < 2,\ x > 0 \}.$$

Clearly A is convex (let $a, b \in A$, then for all $x \in (a,b)$, trivially $x > 0$ and $x^2 < b^2 < 2$, so $x \in A$), but it is not an interval in ~~A as~~ X as A has no supremum in X. (If $X = \mathbb{R}$, we would have $A = (0, \sqrt{2})$.)

(8) ~~TODO →~~ ~~p. 75~~ TODO

~~(9) Proof. Let $\mathcal{J}$ be the dictionary order on on $\mathbb{R}\times\mathbb{R}$~~

~~(9) Proof. Let $\mathcal{J}$ be the dictiona~~

~~(9) Proof. Let $\mathcal{B}$ be the dictionary order basis on $\mathbb{R}\times\mathbb{R}$,~~

$$\cancel{\mathcal{B} = \{(a,b) \mid a < b,\ a,b \in \mathbb{R}\times\mathbb{R}\}},$$

~~and let $\mathcal{B}_d$ be the pro product basis topology on $\mathbb{R}\times\mathbb{R}$ with the discrete and standard topology,~~

$$\cancel{\mathcal{B}_d = \{U \times V \mid U,V \subseteq \mathbb{R},\ V \text{ open w.r.t. standard}\}}.$$

~~We show that $\mathcal{J}(\mathcal{B}) = \mathcal{J}(\mathcal{B}_d)$. "$\subseteq$": Let $x\times y \in \mathbb{R}\times\mathbb{R}$ and let $B \in \mathcal{B}$ with $x\times y \in B$. We have, by definition, $B = (a_1\times a_2, b_1\times b_2)$ for some $a_1,a_2,b_1,b_2 \in \mathbb{R}$ $a_1\times a_2, b_1\times b_2 \in \mathbb{R}\times\mathbb{R}$. Thus, $y\in($ $y \in (a_2,b_2)$. If $a_1 = b_1$, then $B = \{a_1\}\times(a_2,b_2)$, so $x = a_1$ and $y \in (a_2,b_2)$. But the set But then~~

$$\cancel{\mathcal{B}_d \ni \{a_1\}\times(a_2,b_2)}$$

$$\cancel{x\times y \in \{a_1\}\times(a_2,b_2) \subseteq B}$$

~~where $\{a_1\}\times(a_2,b_2) \in \mathcal{B}_d$. If $a_1 \neq b_1$, so $a_1 < b_1$, then $B = (a_1,b_1)\times\mathbb{R} \cup \{a_1\}\times\{x\in\mathbb{R} \mid x > b_1\} \cup \{b_1\}\times\{x\in\mathbb{R} \mid x < b_2\}$. Each set B is composed of is in $\mathcal{B}_d$, too, so so we can show by cases that for some $B_d \in \mathcal{B}_d$, $x \in B_d \subseteq B$, so $\mathcal{J}(\mathcal{B}) \subseteq \mathcal{J}(\mathcal{B}_d)$. "$\supseteq$": Let $x\times y \in \mathbb{R}\times\mathbb{R}$ and let $B_d \in \mathcal{B}_d$ with $x\times y \in B_d$.~~

(5) Proof. Let $\mathcal{B}$ be the dictionary order topology basis on $\mathbb{R} \times \mathbb{R}$ and let $\mathcal{B}_d$ be the basis

$$\cancel{\mathcal{B}_d = \{ U \times V \mid U, V \in \mathbb{R}, \ V \text{ open in standard}}$$

$$\cancel{\mathcal{B}_d = \{ U \times V \mid}$$

$$\mathcal{B}_d = \{ \{ a \} \times (b, c) \mid a, b, c \in \mathbb{R}, \quad b < c \}, \qquad (\ddagger)$$

i.e., the basis of the product topology of the discrete and the standard topology on $\mathbb{R}$. We show that $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}_d)$. "$\subseteq$": We show that $\mathcal{T}(\mathcal{B}_d)$ is finer than $\mathcal{T}(\mathcal{B})$. Let $x \times y \in \mathbb{R} \times \mathbb{R}$ and let $B \in \mathcal{B}$ such that $x \times y \in B$. That is, there are $a_1 \times a_2, b_1 \times b_2 \in \mathbb{R} \times \mathbb{R}$, $a_1 \times a_2 < b_1 \times b_2$, such that ~~$x \times y \in (a_1 \times a_2, b_1 \times b_2)$,~~ $B = (a_1 \times a_2, b_1 \times b_2)$. If $a_1 = b_1$, then $a_2 < b_2$ and

$$B = \{ a_1 \} \times (a_2, b_2).$$

Clearly, $B \in \mathcal{B}_d$, so $\mathcal{T}(\mathcal{B}_d) \supseteq \mathcal{T}\cancel{(\mathcal{B})}(\mathcal{B})$. "$\supseteq$": We show that $\mathcal{T}(\mathcal{B})$ is finer than $\mathcal{T}(\mathcal{B}_d)$. Let $x \times y \in \mathbb{R} \times \mathbb{R}$ and let $B_d \in \mathcal{T}(\mathcal{B}_d)$ such that $x \times y \in B_d$. That is, there are $a_1, a_2, b_2 \in \mathbb{R}$, $a_2 < b_2$, such that $B_d = \{ a_1 \} \times (a_2, b_2)$. But then

$$B_d = \{ a_1 \} \times (a_2, b_2) = (a_1 \times a_2, \ a_1 \times b_2),$$

which is clearly in $\mathcal{B}$. Thus, ~~$\mathcal{T}(\mathcal{B}_d) \supseteq \mathcal{T}(\mathcal{B})$~~ $\mathcal{T}(\mathcal{B})$ is finer than $\mathcal{T}(\mathcal{B}_d)$, too.

$\square$

The standard topology on $\mathbb{R}^2$ is strictly coarser than the above topology.

Proof. Let $\mathcal{B}$ be the standard topology basis on $\mathbb{R}^2$ and let $\mathcal{B}_d$ be defined according to $(\ddagger)$. We show that $\mathcal{T}(\mathcal{B}_d) \supseteq \mathcal{T}(\mathcal{B})$. Let $x \times y \in \mathbb{R} \times \mathbb{R}$ and choose $B \in \mathcal{B}$ such that $x \times y \in B$. That is, choose $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that $x \times y \in (a_1, b_1) \times (a_2, b_2)$. But then

$$\{ x \} \times (b_1, b_2) \subseteq B$$

contains $x \times y$ and is an element of $\mathcal{B}_d$, so $\mathcal{T}(\mathcal{B}_d)$ is finer than $\mathcal{T}(\mathcal{B})$. Now we show that the converse does ~~not~~ not hold. Consider 0×0 and the basis element ~~$\{0\} \times (-1, 1) \in \mathcal{B}_d$.~~ $B_d = \{0\} \times (-1, 1) \in \mathcal{B}_d$. Then there is no $B \in \mathcal{B}$ with $B \subseteq B_d$ as it will always contain elements (x, y) with $x \neq 0$. Thus, $\mathcal{T}(\mathcal{B}_d)$ is <u>strictly</u> finer than $\mathcal{T}(\mathcal{B})$.

$\square$

(10) ~~We use the following notation notation:~~

- ~~$\mathcal{T}_p$ the product topology on $\mathbb{R}\times\mathbb{R}$ $I\times I$~~
- ~~$\mathcal{T}_d$ the dictionary order topology on $I\times I$~~
- ~~$\mathcal{T}_d'$ the topology $I\times I$ inherits as a subspace of $\mathbb{R}\times\mathbb{R}$ in the dictionary order~~

~~Claim: $\mathcal{T}_p$ and $\mathcal{T}_d$ are not comparable.~~

~~Proof. Let $\mathcal{I}_p$ and $\mathcal{I}_d$ be the respective bases.~~

~~$\mathcal{I}_p = \{(a_1, b_1)\times(a_2, b_2) \mid a_1 \times a_2, b_1 \times b_2 \in I\times I\}$,~~

~~$\mathcal{I}_d = \{\{a_1\}\times(a_2, b_2) \mid a_1 \in I, a_1, a_2, b_2 \in I\}$.~~

~~Claim: $\mathcal{T}_d$ is strictly finer than $\mathcal{T}_p$.~~

~~Proof. We show that $\mathcal{T}_d \supseteq \mathcal{T}_p$. Let $\mathcal{B}_d$ and $\mathcal{B}_p$ be the respective bases. Let $x\times y \in \mathbb{R}^2$ $I\times I$ and let $B_d \in$ $B_p \in \mathcal{B}_p$ with $x\times y \in B_p$. That is, there are $a_1, a_2, b_1, b_2 \in \mathbb{R}$ I such that* $B_p = (a_1, b_1)\times(a_2, b_2)$. Now we construct $B_d = \{x\}\times(a_2, b_2)$. Clearly $x\times y \in B_d$, $B_d \subseteq B_p$, and $B_d \in \mathcal{B}_d$. Thus, $\mathcal{T}_d \supseteq \mathcal{T}_p$. Now we show that this is strict. Consider $1/2\times 1/2 \in I\times I$ and $\{1/2\}\times(0, 1) \in \mathcal{B}_d$. Clearly there is no $B_p \in \mathcal{B}_p$ with $1/2\times 1/2$, $B_p \in B_p \in$ that is a subset of $\{1/2\}\times(0, 1)$ as $\{1/2\}$ is a singleton. Thus, $\mathcal{T}_d$ is strictly finer than $\mathcal{T}_p$.~~

$\square$

~~Claim:~~

* ~~One Or with intervals being closed left/right.~~

~~(10) We use the following notation: (with $I^2 = I \times I$, $I = [0,1]$)~~

~~$\mathcal{T}_p$ for the product topology on I^2 with basis~~

~~$\mathcal{B}_p = \{ (a_1, b_1) \times (a_2, b_2) \mid a_1, b_1, a_2, b_2 \in I \}$~~

~~$\cup \{ [0, b_1) \times [a_2, b_2$~~

(10) We use the following notation:

$\mathcal{T}_1$ for the order topology on I with basis

$$\mathcal{B}_1 = \{ (a,b) \mid a, b \in I \} \cup \{ [0,b) \mid b \in I \} \cup \{ (a,1] \mid a \in I \}$$

$\mathcal{T}_p$ for the product topology on $I \times I$ with basis

$$\mathcal{B}_p = \{ B_1 \times B_1' \mid B_1, B_1' \in \mathcal{B}_1 \}$$

$\mathcal{T}_d$ for the dictionary order on I with basis

$$\mathcal{B}_d = \{ (a_1 \times a_2, b_1 \times b_2) \mid a_1 \times a_2, b_1 \times b_2 \in I \times I \}$$

$\mathcal{T}_\mathbb{R}$ for the topology inherited from $\mathbb{R} \times \mathbb{R}$ under the dictionary order with basis

$$\mathcal{B}_\mathbb{R} = \{ (I \times I) \cap (a_1 \times a_2, b_1 \times b_2) \mid a_1 \times a_2, b_1 \times b_2 \in \mathbb{R} \times \mathbb{R} \}$$

$\mathcal{T}_d$ is strictly finer than $\mathcal{T}_\mathbb{R}$

$\mathcal{T}_p$ and $\mathcal{T}_d$ are not ~~com~~ comparable

$\mathcal{T}_p$ and $\mathcal{T}_\mathbb{R}$ are not comparable

(5) (a) Proof. ~~Let X be a~~ Let X and Y be nonempty sets with topologies $\mathcal{T}, \mathcal{T}'$ and $\mathcal{U}, \mathcal{U}'$, respectively. (We denote by X the topological space $(X, \mathcal{T})$ and ~~by by~~ by X' the topological space $(X, \mathcal{T}')$; analogous for Y and Y'.) Suppose $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$ and consider the bases

$$\mathcal{B} = \{ T \times U \mid T \in \mathcal{T},\ U \in \mathcal{U} \} \quad \text{and}$$

$$\mathcal{B}' = \{ T' \times U' \mid T' \in \mathcal{T}',\ U' \in \mathcal{U}' \}$$

of the product topologies $X \times Y$ and $X' \times Y'$, respectively. We show that $\mathcal{T}(\mathcal{B}') \supseteq \mathcal{T}(\mathcal{B})$. Let $x \times y$ be an element of the set $X \times Y$ and let $B \in \mathcal{B}$ such that $x \times y \in B$. As $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$, also $\mathcal{B}' \supseteq \mathcal{B}$, so $B \in \mathcal{B}'$. Hence, $\mathcal{T}(\mathcal{B}')$ is finer than $\mathcal{T}(\mathcal{B})$.

$\square$

~~(b) W.l.o.g. Consider Y.~~

~~Proof. Let X and Y be nonempty sets with topologies $\mathcal{T}, \mathcal{T}'$ and $\mathcal{U}, \mathcal{U}'$, respectively. Suppose, for contraposition, that either $\mathcal{T}' \subset \mathcal{T}$ or that either not $\mathcal{T}' \supseteq \mathcal{T}$ or not $\mathcal{U}' \supseteq \mathcal{U}$. W.l.o.g., suppose that not $\mathcal{T}' \supseteq \mathcal{T}$ (i.e., there is some $T \in \mathcal{T}$ with $T \notin \mathcal{T}'$). Consider the bases $\mathcal{B}, \mathcal{B}'$ as defined above. We show that not $\mathcal{T}(\mathcal{B}') \supseteq \mathcal{T}(\mathcal{B})$. Suppose, for contradiction, that $\mathcal{T}(\mathcal{B}') \supseteq \mathcal{T}(\mathcal{B})$. Let $V \in \mathcal{T}$ with $V \notin \mathcal{T}'$. Let $x \in V$ and choose $B \in \mathcal{B}$ with $x \in B$ and $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.~~

(b) Proof. Let $(X, \mathcal{T}), (X, \mathcal{T}'), (Y, \mathcal{U}), (Y, \mathcal{U}')$ be topological spaces ~~over~~ over X, Y denoted by X, X', Y, Y', ~~re~~ respectively. Let $\mathcal{B}, \mathcal{B}', \mathcal{C}, \mathcal{C}'$ be bases for them, respectively. Then the collections

~~$\mathcal{D} = \{ U \times V \mid U \in \mathcal{T},\ V \in \mathcal{U} \}$~~ ~~and~~

~~$\mathcal{D}' = \{ U \times V \mid U \in \mathcal{T}',\ V \in \mathcal{U}' \}$~~

are bases for the product topologies ~~of~~ $X \times Y$ and $X' \times Y'$, respectively. Suppose that $\mathcal{T}(\mathcal{B}') \supseteq \mathcal{T}(\mathcal{B})$. We show that $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$. Let $x \in X$ and $y \in Y$. Choose $B \in \mathcal{B}$ such that $x \in B$ and choose $C \in \mathcal{C}$ such that $y \in C$. Then $x \times y \in B \times C \in \mathcal{D}$. As $\mathcal{T}(\mathcal{D}')$ is finer than $\mathcal{T}(\mathcal{D})$, there is an $B' \times C' \in \mathcal{D}'$ such that $x \times y \in B' \times C' \subseteq B \times C$. That is, $x \in B' \subseteq B$ and $y \in C' \subseteq C$ where $B' \in \mathcal{B}'$ and $C' \in \mathcal{C}'$. Hence, $\mathcal{T}' \supseteq \mathcal{T}$ and $\mathcal{U}' \supseteq \mathcal{U}$.

$\square$

17 Closed Sets and Limit Points

Definition (Closed Set): Let X be a topological space. A set $A \subseteq X$ is <u>closed</u> if $X \setminus A$ is open.

Theorem ("Closed" Topology): Let X be a topological space. Then:

- $\emptyset$ and X are closed;
- arbitrary intersections of closed sets are closed;
- finite unions of closed sets are closed.

Theorem (Closed Sets in Subspaces): Let Y be a subspace of X. A set ~~$A \subseteq Y$~~ $A \subseteq Y$ is closed in Y if and only if there is a closed set $B \subseteq X$ such that $A = Y \cap B$.

Theorem (Closed Sets in Subspaces): Let Y be a subspace of X. If A is closed in X and Y is closed in X, then A is closed in X.

Definition (Closure and Interior): Let X be a topological space and let $A \subseteq X$. Then the <u>closure</u> $\mathrm{cl}\,A$ of A is the intersection of all closed sets containing A and the <u>interior</u> $\mathrm{int}\,A$ of A is the union of all open sets contained in A. If A is open, $\mathrm{int}\,A = A$ and if A is closed, $\mathrm{cl}\,A = A$.

Theorem (Closure in Subspaces): Let Y be a subspace of X ~~an~~ and let $A \subseteq Y$ and let $\bar{A}$ be the closure of A in X. Then the closure of A in Y is $Y \cap \bar{A}$.

Theorem (Closure via Basis): Let X be a topological space and let $A \subseteq X$. Then:

(i) $x \in \bar{A}$ if and only if every open set U with ~~$x \in$~~ $x \in U$ intersects A, i.e., $U \cap A \neq \emptyset$;

(ii) Let $\mathcal{B}$ be a basis for the topology on X, then $x \in \bar{A}$ if and only if every basis element $B \in \mathcal{B}$ with $x \in B$ intersects A.

Definition (Limit Point): Let X be a topological space and let $A \subseteq X$. A point $x \in X$ is a <u>limit point</u> of A if every ~~neighbo~~ neighborhood of x, i.e., every open set containing x, intersects A at some point other than x. That is, x is a limit point if ~~$x \in$~~ $x \in \mathrm{cl}(A \setminus \{x\})$. (It is not required that $x \in A$!)

Theorem (Closure and Limit Points): Let A be a subset of a topological space X. Let A' be the set of all limit points of A. Then $\bar{A} = A \cup A'$.

Corollary: A subset of a topological space is closed if and only if it contains all its limit points.

Definition (Hausdorff Space): A topological space X is called a <u>Hausdorff space</u> if for all distinct $x_1, x_2 \in X$, there exist neighborhoods U_1 and U_2, respectively, that are disjoint.

Theorem: Let X be a Hausdorff space. Then every finite $A \subseteq X$ is closed.

Theorem: Let X be a topological space satisfying the T_1 axiom. Let $A \subseteq X$. Then $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.

Theorem: If X is a Hausdorff space, then a sequence $(x_n) \subseteq X$ converges to at most one point in X.

Definition (Convergence): Let X be a topological space and let $(x_n) \subseteq X$ be a sequence. Then (x_n) converges to some $x \in X$, say $x_n \to x$, if for all neighborhoods U of x there is an $N \in \mathbb{Z}_+$ such that $x_n \in U$ for all $n \geq N$.

~~Theorem (Order Topology/Hausdorff space)~~

Theorem: Every simply ordered set is a Hausdorff space in the order topology. The product of two Hausdorff spaces is a Hausdorff space. A subspace of a Hausdorff space is a Hausdorff space.

Exercises:

(1) Proof. Let $\mathcal{C}$ and $\mathcal{J}$ be defined as given. We check the axioms of a topology one-by-one.

(i) Clearly $\emptyset, X \in \mathcal{J}$ as $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$.

(ii) Let $\{U_\alpha\} \subseteq \mathcal{J}$. Then

$$X \setminus \bigcup U_\alpha = \bigcap (X \setminus U_\alpha) \in \mathcal{C}$$

as $X \setminus U_\alpha \in \mathcal{C}$ by definition. Thus, $\bigcup U_\alpha \in \mathcal{J}$.

(iii) Let $U_1, \dots, U_k \in \mathcal{J}$. Then

$$X \setminus (U_1 \cap \dots \cap U_k) = (X \setminus U_1) \cup \dots \cup (X \setminus U_k) \in \mathcal{C}$$

as $X \setminus U_i \in \mathcal{C}$ by definition. Thus, $U_1 \cap \dots \cap U_k \in \mathcal{J}$. $\square$

(2) Proof. Let Y be a subspace of X and let $A \subseteq Y$ be closed in Y. Suppose Y is closed in X. Then there is a set $K \subseteq X$ closed in X such that $A = Y \cap K$. However, as Y is closed in X, this means that A is closed in X as well.

□

(3) Proof. Let X and Y be topological ~~spaces~~ spaces and let $A \subseteq X$ and $B \subseteq Y$ be closed. Consider $A \times B \subseteq X \times Y$. As A and B are closed, $X \setminus A$ and $Y \setminus B$ are open and thus, $(X \setminus A) \times (Y \setminus B)$ are open in $X \times Y$, too. However,

$$(X \setminus A) \times (Y \setminus B) = (X \times Y) \setminus (A \times B),$$

so $A \times B$ is closed in $X \times Y$.

□

~~(4) Proof. Let U and A be open and closed in X, respectively. Consider $U \setminus A$ and $A \setminus U$.~~

(4) Proof. Let U be ~~closed~~ open and let A be ~~open~~ closed in X. Then $A \setminus X$ is open and as $U \setminus A = U \cap (X \setminus A)$, $U \setminus A$ is open, too. ~~As~~ As U is open, $X \setminus U$ is closed and thus we have that $A \setminus U = A \cap (X \setminus U)$ is closed as well.

□

~~(5) Proof. Let X be an ordered set as with the order topology. Let $a, b \in X$, $a < b$. Set $A = (a,b)$ and let $x \in \bar{A}$. Let $\mathcal{B}$ be the order topology basis. If a and b are not the minimum and maximum of X, respectively, then $[a,b]$ is closed, so $\bar{A} \subseteq [a,b]$.~~

(5) Proof. Let X be an ordered set with the order topology. Let $a, b \in X$, $a < b$. If a and b are the ~~minima~~ minimum and maximum, respectively, (a,b) is closed. If a is the minimum and b is not the ~~max~~ ~~max~~ maximum, $(a,b]$ is closed. If a is not the minimum and b is the maximum, $[a,b)$ is closed. If neither a nor b is the minimum or maximum, respectively, then $[a,b]$ is closed. Thus, $\overline{(a,b)} \subseteq [a,b]$ with equality iff a and b are not extreme.

□

~~(6) Proof~~

(6) (a) Proof. Let X be a topological space and let $A \subseteq B \subseteq X$. We show that $\overline{A} \subseteq \overline{B}$. Let $x \in \overline{A}$. Then for all open $U \subseteq X$ ~~as~~ with $x \in U$ we have $A \cap U \neq \emptyset$. But as $A \subseteq B$, also $B \cap U \neq \emptyset$, so $x \in \overline{B}$, too.

$\square$

(b) Proof. Let X be a topological space and let $A, B \subseteq X$. We ~~show~~ show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. "$\subseteq$": Let $x \in \overline{A \cup B}$. Then for all open $U \subseteq X$ with $x \in U$, $(A \cup B) \cap U \neq \emptyset$. But then $A \cap U \neq \emptyset$ or $B \cap U \neq \emptyset$, so $x \in \overline{A}$ or $x \in \overline{B}$, and thus $x \in \overline{A} \cup \overline{B}$. "$\supseteq$": Let $x \in \overline{A} \cup \overline{B}$. Suppose $x \in \overline{A}$. Then for all ~~U~~ open $U \subseteq X$ with $x \in U$, $A \cap U \neq \emptyset$. But then also $(A \cup B) \cap U \neq \emptyset$, so $x \in \overline{A \cup B}$. The case for $x \in B$ is analogous.

$\square$

(c) Proof. Let X be a topological space and let $\{A_\alpha\}$ be a collection of subsets of X. We show

$$\overline{\bigcup A_\alpha} \supseteq \bigcup \overline{A_\alpha}.$$

Let $x \in \bigcup \overline{A_\alpha}$. ~~Then for all U open U⊆X with x∈U,~~ Then there is some A_α such that for all ~~U~~ open $U \subseteq X$ with $x \in U$, $A_\alpha \cap U \neq \emptyset$. Thus, also $(\bigcup A_\alpha) \cap U \neq \emptyset$, so $x \in \overline{\bigcup A_\alpha}$.

$\square$

Note that the converse does not hold. Consider $X = \mathbb{R}$ and the collection $\{A_n\}_{n \in \mathbb{Z}_+}$ with $A_n = \{1/n\}$. Then

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{\{1/n \mid n \in \mathbb{Z}_+\}} = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}$$

but $\overline{A_n} = A_n$ for all $n \in \mathbb{Z}_+$, so $\bigcup_n \overline{A_n} = \bigcup_n A_n$.

(7) The step "Thus U must intersect some A_α [...]" does not hold for all U, so x is not necessarily in $\overline{A_\alpha}$ as not all U intersect with it. This is not a problem in the finite case as there are not "enough" sets to "run away" with infinitely many intersections.

7 2✓
6 2

(8) (a) "$\subseteq$": Proof. Let $x \in \overline{A \cap B}$, then all open $U \subseteq X$ with $x \in U$ intersect $A \cap B$. But then also $U \cap A \neq \emptyset$ and $U \cap B \neq \emptyset$, so $x \in \bar{A}$ and $x \in \bar{B}$. Thus, $x \in \bar{A} \cap \bar{B}$, so $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.

$\square$

"$\supseteq$": False. Consider $X = \mathbb{R}$ and $A = (-\infty, 0)$ and $B = (0, \infty)$. Then $\bar{A} = (-\infty, 0]$ and $\bar{B} = [0, \infty)$, so $\bar{A} \cap \bar{B} = \{0\}$, but $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset \neq \{0\}$.

(b) "$\subseteq$": Proof. Let $x \in \overline{\bigcap A_\alpha}$, then for all open $U \subseteq X$ ~~and~~ with $x \in U$, ~~$U \cap A_\alpha \neq \emptyset$~~ $U \cap (\bigcap A_\alpha) \neq \emptyset$. Thus, for all A_α, $U \cap A_\alpha \neq \emptyset$, so $x \in \bar{A}_\alpha$. Therefore, also $x \in \bigcap \bar{A}_\alpha$, so $\overline{\bigcap A_\alpha} \subseteq \bigcap \bar{A}_\alpha$.

$\square$

"$\supseteq$": False. See (a) ~~fo~~ for a counterexample.

(c) "$\subseteq$": False. Consider $X = \mathbb{R}$, $A = (-\infty, 0)$, and $B = \{0\}$. Then $\bar{A} = (-\infty, 0]$ and $\bar{B} = \{0\}$. ~~Moreover,~~ $A \setminus B = A$ and $\bar{A} \setminus \bar{B} = (-\infty, 0) \neq (-\infty, 0] = \bar{A} = \overline{A \setminus B}$.

"$\supseteq$": Proof. Let $x \in \bar{A} \setminus \bar{B}$. Then for all open $U \subseteq X$ ~~w~~ with $x \in A$, $U \cap A \neq \emptyset$ but $U \cap B = \emptyset$. ~~Thus~~ Therefore,

$$U \cap (A \setminus B) = (U \cap A) \setminus (U \cap B) = U \cap A \neq \emptyset,$$

so $x \in \overline{A \setminus B}$, too.

$\square$

(9) Proof. Let X and Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$. We show that in $X \times Y$, $\overline{A \times B} = \bar{A} \times \bar{B}$.
~~"$\subseteq$": Let $x \times y \in \overline{A \times B}$. Then for all open $U \times V \subseteq X \times Y$, $(U \times V) \cap (A \times B) \neq \emptyset$. Therefore, $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$.~~
"$\subseteq$": Let $x \times y \in \overline{A \times B}$. Then for all open $U \subseteq X$ and for all open $V \subseteq Y$ ~~wi~~ with $x \times y \in U \times V$, i.e., all basis elements of $X \times Y$ containing $x \times y$, we have $(U \times V) \cap (A \times B) \neq \emptyset$. That is, $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$, so $x \in \bar{A}$ and $y \in \bar{B}$. Hence, $x \times y \in \bar{A} \times \bar{B}$. "$\supseteq$": Let ~~$x \in$~~ $x \times y \in \bar{A} \times \bar{B}$. Then for all open ~~$U \subseteq$~~ $U \subseteq X$ and all open $V \subseteq Y$ with $x \in U$ and $y \in V$, we have $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. ~~As~~ Thus, $x \times y \in U \times V$ and $(U \times V) \cap (A \times B) \neq \emptyset$ and as the collection of products of open sets constitutes a basis for $X \times Y$, $x \times y \in \overline{A \times B}$.

$\square$

~~(10) Proof. Let X be a simply ordered set and let $\mathcal{T}$ be the order topology on X. Let $a, b \in X$, $a \neq b$, and suppose that $a < b$. If b is the immediate successor If b is not the immediate successor of a, there is some some $c \in (a, b)$.~~

(10) Proof. Let X be a simply ordered set with the order topology. Let $a, b \in X$, $a \neq b$. Suppose w.l.o.g. that $a < b$. If a is the smallest element of X and b is the largest element, both $[a, b)$ and $(a, b]$ are open, ~~and~~ contain a and b, respectively, and are disjoint. If a is the smallest element and b is not the largest element, there is a $c > b$. Then $[a, b)$ and (a, c) contain a and b, respectively, are open, and disjoint. If a is not the smallest element and b is the largest element, there is a $c < a$. Then (c, b) and $(a, b]$ are open, disjoint, and contain a and b, ~~re~~ respectively. If a and b are ~~both~~ not ~~the~~ the ~~men~~ smallest/largest element, there are $c, c' \in X$ with $c < a$ and $c' > b$. Then (c, b) and (a, c') contain a and b, respectively, are disjoint, and open. ~~Then~~ Thus, X is Hausdorff.

$\square$

(11) Proof. Let X and Y be Hausdorff ~~spa~~ spaces. Consider the product topology on $X \times Y$ and let ~~$x \times y$~~ $x \times y, x' \times y' \in X \times Y$ with $x \times y \neq x' \times y'$. Suppose $x \neq x'$ and let $U, U' \subseteq X$ be open such that $x \in U$, $x' \in U'$, and $U \cap U' = \emptyset$. Let $V, V' \subseteq Y$ be open such that $y \in V$ and $y' \in V'$. (Note that V and V' are not necessarily disjoint.) Then $x \times y \in U \times V$ and $x' \times y' \in U' \times V'$ where $U \times V$ and $U' \times V'$ are open. As $U \cap U' = \emptyset$, also $(U \times V) \cap (U' \times V') = \emptyset$. If $x = x'$ and $y \neq y'$, the argument is analogous. Thus, $X \times Y$ is Hausdorff.

$\square$

(12) Proof. Let X be a Hausdorff space and let $Y \subseteq X$ be equipped with the inherited subspace topology. Let $a, b \in Y$, $a \neq b$. Then there are open sets $U, V \subseteq X$ with $a \in U$ and $b \in V$ such that $U \cap V = \emptyset$. But then $U' = Y \cap U$ and $V' = Y \cap V$ are open in Y, are disjoint, and contain a and b, respectively. Thus, Y is also a Hausdorff space.

$\square$

(13) Proof. Let X be a topological space. "Only if:" Suppose that X is Hausdorff. Then $X \times X$ is ~~Haus~~ Hausdorff, too. Consider $\Delta = \{x \times x \mid x \in X\}$. We show that $\Delta' \subseteq \Delta$, where Δ' denotes all limit points of Δ. Let $x \times y \in \Delta'$ and suppose, for contradiction, that $x \neq y$, i.e., $x \times y \notin \Delta$. Then there are disjoint open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$. But then $(U \times V) \cap \Delta = \emptyset$. ↯ This is a contradiction as $x \times y$ was assumed to be a limit point. Hence, $x = y$, so $\Delta' \subseteq \Delta$. That is, Δ is closed. "If:" ~~Contraposition.~~ Suppose that X is not Hausdorff. We show that $\Delta = \{x \times x \mid x \in X\}$ is not closed in $X \times X$. ~~Let $x, y \in X$, $x \neq y$~~ There exist $x, y \in X$, $x \neq y$, such that there are no ~~distinct neighborhoods of~~ open $U, V \subseteq X$ with $x \in U$ and $y \in V$ that are disjoint. We show that not $\Delta' \subseteq \Delta$, i.e. ~~Consider distinct x,y $x, y \in X$ and open sets $U, V \subseteq X$ with $x \in U$ and $y \in V$ such that $U \cap V \neq \emptyset$.~~ Consider the $x, y \in X$, $x \neq y$ such that there are no disjoint neighborhoods of x and y. Let $U, V \subseteq X$ be open sets such that $x \in U$ and $y \in V$. Then $U \cap V \neq \emptyset$, so $(U \times V) \cap \Delta \neq \emptyset$. Thus, $x \times y \in \Delta'$, but as $x \neq y$, $x \times y \notin \Delta$. Therefore, $\Delta' \not\subseteq \Delta$ and thus Δ is not closed. □

(14) ~~Proof.~~ Let ~~$\mathcal{T}_f$~~ $\mathcal{T}_f$ be the finite complement topology on $\mathbb{R}$, i.e.,

$$\mathcal{T}_f = \{ U \subseteq \mathbb{R} \mid \mathbb{R} \setminus U \text{ is finite or all of } \mathbb{R} \}.$$

Consider the sequence $(x_n) \subseteq \mathbb{R}$, $x_n = 1/n$. Then this sequence converges to every $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ and let $U \in \mathcal{T}_f$ with $x \in U$. Then $\mathbb{R} \setminus U$ is finite, so $x_n \in \mathbb{R} \setminus U$ can only hold for finitely many $n \in \mathbb{Z}_+$. Let $N \in \mathbb{N}$ be the largest number such that $x_N \in \mathbb{R} \setminus U$. Then $x_n \in U$ for all ~~$n \geq$~~ $n > N$, so $x_n \to x$. □

~~(15)~~

(15) Claim: Let X be a topological space. Then X fulfills the T_1 axiom, i.e., every finite set is closed, if and only if for each $x, y \in X$, each has a neighborhood in X not containing the other.

~~Proof. Let X be a topological space. "Only if:" Suppose that X fulfills the~~ ~~T_1~~ ~~axiom.~~

~~Proof. Let X be a topological space. "If:" Suppose, for contraposition, that X does not fulfill the~~ ~~T_1~~ ~~axiom. Then there is a finite closed set~~ ~~$C \subseteq X$~~. ~~If X is finite, then~~

TODO

~~(16)~~ ~~(a)~~

(16)

$\mathcal{T}_1$ = standard topology on $\mathbb{R}$

$\mathcal{T}_2$ = topology of $\mathbb{R}_K$

$\mathcal{T}_3$ = finite complement topology on $\mathbb{R}$

$\mathcal{T}_4$ = ~~uppet~~ upper limit topology (with basis $(a, b]$)

$\mathcal{T}_5$ = topology with basis $(-\infty, a)$ on $\mathbb{R}$

(a) $K = \{ 1/n \mid n \in \mathbb{Z}_+ \}$

Closure under $\mathcal{T}_i$...

(1) ~~$\bar{K} = \{0\} \cup K$~~ $\bar{K} = K \cup \{0\}$

(2) $\bar{K} = K$

(3) $\bar{K} = \mathbb{R}$

(4) $\bar{K} = K$

(5) $\bar{K} = K \cup \{0\}$

We prove each closure.

(1) Clearly, 0 is a limit point of K: Let ~~(a,b) be~~ U be a neighborhood of 0. Then there are $a < 0 < b$ such that $(a,b) \subseteq U$. But then there is some $N \in \mathbb{N}$ such that $1/n < b$ for all $n \geq N$. Thus, $(a,b) \cap K$ is infinite and 0 is a limit point of K. ~~So~~ Suppose $x \neq 0$ is a limit point. If $x < 0$, then $(-\infty, 0) \ni x$, but $(-\infty, 0) \cap K = \emptyset$. ↯. If ~~$x > 0$~~, $1 > x > 0$ then there is a $n \in \mathbb{N}$ with $1/(n+1) < x < 1/n$, so $x \in (1/(n+1), n)$, but $(1/(n+1), 1/n) \cap K = \emptyset$ ↯. If $x > 1$, then $(1, \infty) \ni x$, but $(1, \infty) \cap K = \emptyset$. Thus, 0 is the only limit point (not in K). ■

(16) (a) (2) ~~We show that K is closed in $\mathcal{T}_2$. Let $x \in$ ... $x \in \mathbb{R}$ be any ... limit point of K.~~ We directly show that $\overline{K} = K$. "$\supseteq$" is clear. "$\subseteq$": Let $x \in \overline{K}$ and suppose $x \notin K$. But then $x \in (x-1, x+1) \setminus K$, where the set is a basis element, and

$$((x-1, x+1) \setminus K) \cap K = \emptyset. \lightning$$

~~This means x is not a limit point of~~ Hence, $x \in K$, so $\overline{K} = K$. ∎

(3) We directly show $\overline{K} = \mathbb{R}$. "$\subseteq$" is clear. "$\supseteq$": Let $x \in \mathbb{R}$ and let $U \subseteq \mathbb{R}$ be open. ~~Then $\mathbb{R}$~~ If $U = \mathbb{R}$, things are trivial. Suppose $U \neq \mathbb{R}$. Then $\mathbb{R} \setminus U$ is finite, so $(\mathbb{R} \setminus U) \cap K$ is finite. Hence, as K is infinite, $U \cap K$ is nonempty. Thus, $x \in \overline{K}$ and as x was arbitrary, $\mathbb{R} \subseteq \overline{K}$. ∎

(4) "$\supseteq$" is clear. "$\subseteq$": Let $x \in \overline{K}$ and suppose $x \notin K$.
If $x \leq 0$, then $x \in (-\infty, 0]$, but $(-\infty, 0] \cap K = \emptyset$. $\lightning$
If ~~$x \geq$~~ $x > 1$, then $x \in (1, \infty)$, but $(1, \infty) \cap K = \emptyset$. $\lightning$
If $0 < x < 1$, then there is an $n \in \mathbb{N}$ with $1/(n+1) < x < 1/n$, so $x \in (1/(n+1), 1/n)$, but $(1/(n+1), 1/n) \cap K = \emptyset$. $\lightning$ Hence, $\overline{K} \subseteq K$. ∎

(5) Analogous to (1).

(b) $\mathcal{T}_1$ Hausdorff (it is an order topology)

$\mathcal{T}_2$ Hausdorff (finer than $\mathcal{T}_1$)

$\mathcal{T}_3$ not Hausdorff, fulfills T_1

$\mathcal{T}_4$ Hausdorff (finer than $\mathcal{T}_1$)

$\mathcal{T}_5$ not Hausdorff, does ~~not fulf~~ not fulfill T_1

~~(17) Consider on $\mathbb{R}$ the topologies $\mathcal{T}_\ell$ and $\mathcal{T}_c$ with bases~~

~~$\mathcal{B}_\ell = \{[a, b) \mid a, b \in \mathbb{R}, a < b\}$ and~~

~~$\mathcal{B}_c = \{[a, b) \mid a, b \in \mathbb{Q}, a < b\}$,~~

~~respectively. Consider $A = (0, \sqrt{2})$ and $B = (\sqrt{2}, 3)$. Then~~

~~$[0, \sqrt{2})$~~

~~$\overline{A}_\ell = (0, \sqrt{2}]$~~ ~~$\overline{A}_c = [0, \sqrt{2}]$~~ ~~$[0, \sqrt{2}]$~~

~~$\overline{B}_\ell = (\sqrt{2}, 3]$~~ ~~$\overline{B}_c =$~~

~~$[\sqrt{2}, 3)$~~

(17) Consider the topologies on $\mathbb{R}$ given ~~by~~ by the bases

$$\mathcal{B} = \{[a,b) \mid a,b \in \mathbb{R}\} \quad \text{and}$$

$$\mathcal{C} = \{[a,b) \mid a,b \in \mathbb{Q}\}.$$

Consider the sets

$$A = (0, \sqrt{2}) \quad \text{and} \quad B = (\sqrt{2}, 3).$$

W.r.t. $\mathcal{T}(\mathcal{B})$, we have:

$\bar{A} = [0, \sqrt{2})$ We have $\mathbb{R} \setminus \bar{A} = (-\infty, 0) \cup [\sqrt{2}, \infty)$, which is open. However, $\mathbb{R} \setminus (0, \sqrt{2}) = (-\infty, 0] \cup [\sqrt{2}, \infty)$ would not be open.

$\bar{B} = [\sqrt{2}, 3)$ Same reasoning.

W.r.t. $\mathcal{T}(\mathcal{C})$, we have:

$\bar{A} = [0, \sqrt{2}]$ Proof. Clearly, $0 \in \bar{A}$ as all $[a,b) \in \mathcal{C}$ with $a \leq 0 < b$, $A \cap [a,b) \neq \emptyset$ as $b > 0$. ~~Moreover~~ Moreover, $\sqrt{2} \in \bar{A}$ as for all $[a,b) \in \mathcal{C}$ with $\sqrt{2} \in [a,b)$, we have $a < \sqrt{2}$ as a must be rational. Hence, $[a,b) \cap A \neq \emptyset$. Now let $x < 0$. Then $x \in [x, 0)$, but ~~$[x,0) \cap A = \emptyset$. Let $x > \sqrt{2}$. Then there~~ $(-\infty, 0) \cap A = \emptyset$. Let $x > \sqrt{2}$. Then there is a rational a with $\sqrt{2} < a < x$, so $x \in [a, \infty)$, but $[a, \infty) \cap A = \emptyset$. ■

~~$\bar{B} = [\sqrt{2}, 3]$ Proof. Clearly, $3 \in \bar{B}$ as~~

$\bar{B} = [\sqrt{2}, 3)$ Proof. Clearly, $3 \notin \bar{B}$ as $3 \in [3, \infty)$, but $[3, \infty) \cap B = \emptyset$. We have $\sqrt{2} \in \bar{B}$ as for all $[a,b) \in \mathcal{C}$ with $\sqrt{2} \in [a,b)$, ~~$b<$~~ $b > \sqrt{2}$, so $[a,b) \cap B \neq \emptyset$. ■

(18) (a) $A = \{ \frac{1}{n} \times 0 \mid n \in \mathbb{Z}_+ \}$

$\bar{A} = A \cup \{ 0 \times 1 \}$

Proof. ~~Let $x \times y \in \bar{A}$ and suppose~~ Let $x \times y \in \bar{A} \setminus A$. ~~Then for all basis~~ If $x = y = 0$, then $[0 \times 0, 0 \times 1)$ contains $\cancel{0\times}$ $x \times y$, is a basis element, but $[0 \times 0, 0 \times 1) \cap A = \emptyset$. ↯ Thus, $x \times y \notin \bar{A}$. If $x = 0$ and $\cancel{y \neq 1}$ $0 < y < 1$, then $x \times y \in (0 \times 0, 0 \times 1)$, which is a basis element, but $\cancel{[\times 0}$ $(0 \times 0, 0 \times 1) \cap A = \emptyset$. ↯ Thus, $x \times y \notin \bar{A}$. If $x \neq 0$ (~~and y arbitrary~~) ~~th~~ there is an $n \in \mathbb{Z}_+$ such that $1/(n+1) < x < 1/n$. Moreover, there are $a, b \in (0,1)$ such that $1/(n+1) < a < x < b < 1/n$. Then $x \times y \in (a \times 0, b \times 1)$, but $(a \times 0, b \times 1) \cap A = \emptyset$. ↯ If $x = 0$ and $y = 1$, let B be any basis element such that $x \times y \in B$. Then B has the form $[0 \times 0, b_1 \times b_2)$ or $(0 \times a_2, b_1 \times b_2)$ for $\cancel{0 < b_1}$ $\cancel{0 < b_1 \leq 1}$ $b_1 > 0$ and arbitrary $a_2, b_2 \in I$. As the latter is a subset of the former, we consider only $(0 \times a_2, b_1 \times b_2)$. Clearly, $x \times y = 0 \times 1 \in (0 \times a_2, b_1 \times b_2)$. Moreover, $(0 \times a_2, b_1 \times b_2) \cap A = (0 \times 0, b_1 \times 0) \neq \emptyset$. Thus, $\cancel{x \times y}$ $0 \times 1 \in \bar{A}$. □

(b) $B = \{ (1 - 1/n) \times 1/2 \mid n \in \mathbb{Z}_+ \}$

$\bar{B} =$

TODO

(13) (a) Proof. Let X be a topological space and let $A \subseteq X$. We show that $(\operatorname{int} A) \cap (\operatorname{bd} A) = \emptyset$. Let $x \in \operatorname{int} A$. Then there is a neighborhood $U \subseteq X$ of x such that $U \subseteq A$. Suppose, for contradiction, that ~~$x \in \operatorname{bd} A$~~ $x \in \operatorname{bd} A$. Then $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. That is, for all neighborhoods $V \subseteq X$ of x, $V \cap (X \setminus A) = V \setminus A \neq \emptyset$. ↯ As U is a neighborhood of x and $U \subseteq A$, $U \setminus A = \emptyset$. Hence, $x \notin \operatorname{bd} A$. Conversely, let $x \in \operatorname{bd} A$. Then $x \in \overline{A}$ and $x \in \overline{X \setminus A}$. Let U be a neighborhood of x. Then $U \cap (X \setminus A) = U \setminus A \neq \emptyset$. Hence, $U \not\subseteq A$, so $x \notin \operatorname{int} A$. Thus, $\operatorname{int} A$ and $\operatorname{bd} A$ are ~~d~~ disjoint.

□

Proof. Let X be a topological space and let $A \subseteq X$. We show that $\overline{A} = (\operatorname{int} A) \cup (\operatorname{bd} A)$. ~~"⊆": Let $x \in \overline{A}$. "⊇": Let $x \in \operatorname{int} A$. As $\operatorname{int} A \subseteq A \subseteq \overline{A}$, $x \in \overline{A}$. Let $x \in \operatorname{bd} A$.~~ "⊇" is clear as $\operatorname{int} A \subseteq A \subseteq \overline{A}$ and $\operatorname{bd} A = \overline{A} \cap \overline{X \setminus A} \subseteq \overline{A}$. "⊆": Let $x \in \overline{A}$. If there is a neighborhood $U \subseteq X$ of x such that $U \subseteq A$, $x \in \operatorname{int} A$. Suppose there is no ~~so~~ such neighborhood. Still, for all neighborhoods $U \subseteq X$ of x, $A \cap U \neq \emptyset$. But as $U \not\subseteq A$, also $(X \setminus A) \cap U \neq \emptyset$, so $x \in \overline{X \setminus A}$. Hence, $x \in \operatorname{bd} A$.

□

(b) Proof. Let X be a topological space and let $A \subseteq X$.
~~"⇒": Suppose $A = \emptyset$. $\operatorname{bd} A = \emptyset$. Then $\overline{A} = \operatorname{int} A$.~~
"⇐": Suppose A is open and closed. That is, $A = \operatorname{int} A$ and $A = \overline{A}$. But then $\overline{A} = \operatorname{int} A$, so $\operatorname{bd} A = \emptyset$.
"⇐": Suppose $\operatorname{bd} A = \emptyset$. Then $\overline{A} = \operatorname{int} A$. We show that A is ~~open~~ closed and ~~closed~~ open, i.e., $A = \overline{A}$ and $A = \operatorname{int} A$, respectively. First, $A = \overline{A}$. "⊆": Trivial. ~~"⊇": Let $x \in \overline{A}$. As $\overline{A} \subseteq \operatorname{int} A$, for $\overline{A} = \operatorname{int} A$, for all neighbor~~
"⊇": As $\overline{A} = \operatorname{int} A \subseteq A$, this is clear. Second, $A = \operatorname{int} A$. "⊇": Trivial. "⊆": As $A \subseteq \overline{A} = \operatorname{int} A$, this is clear. Hence, A is closed and open.

□

(c) Proof. Let X be a topological space and let $U \subseteq X$. "⇒": Suppose U is open. Then $U = \operatorname{int} U$. As $\overline{U} = \operatorname{int} U \cup \operatorname{bd} U$, we have $\operatorname{bd} U = \overline{U} \setminus U$. "⇐" Suppose U is not open. Then there is some $x \in U$ with $x \notin \operatorname{int} U$. But as $\overline{U} = \operatorname{int} U \cup \operatorname{bd} U$ and $U \subseteq \overline{U}$, this means that $x \in \operatorname{bd} U$. However, $x \notin \overline{U} \setminus U$, showing the claim by contraposition.

□

~~(d) Yes. Proof. Let X be a topological space and let $U \subseteq X$ be open. Then $\overline{U} = \overline{U} = \operatorname{int} \overline{U} \cup \operatorname{bd} \overline{U}$. Thus, $\operatorname{int} \overline{U} = \overline{U} \setminus \operatorname{bd} \overline{U}$. Plugging in the the definition of $\operatorname{bd} \overline{U}$ and using $\overline{\overline{U}} = \overline{U}$,~~

~~$\operatorname{int} \overline{U} = \overline{U} \setminus \operatorname{bd} \overline{U} = \overline{U} \setminus (\overline{U} \cap \overline{X \setminus \overline{U}}) = \overline{U} \setminus \overline{X \setminus \overline{U}}$~~

~~(19) (d) No. Consider the lower-limit topology $\mathbb{R}_\ell$ and the set $U=[0,1)$. Clearly, U is open.~~

(19) (d) No. Consider $X=\{a,b,c\}$ with the topology

$$\mathcal{T}=\{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}\}.$$

Consider the open $\{a\}\in\mathcal{T}$. Then $\overline{\{a\}}=\{a,b\}$, but this set is itself open, so ~~$\operatorname{int}\{a,b\}$~~

$$\operatorname{int}\overline{\{a\}}=\{a,b\}\neq\{a\}.$$

Definition (Boundary): Let X be a topological space and let $A\subseteq X$. Then the boundary of A is

$$\operatorname{bd} A=\overline{A}\cap\overline{X\setminus A}.$$

We have $\operatorname{int}A\cap\operatorname{bd}A=\emptyset$ and $\overline{A}=\operatorname{int}A\cup\operatorname{bd}A$.

(20) (a) $A=\{x\times y \mid y=0\}$

$\operatorname{int} A=\emptyset$

$\operatorname{bd} A=A$

Proof. We show that $A=\overline{A}$ and $\operatorname{int}A=\emptyset$. ~~Let $x\in\overline{A}$.~~ We have

$$\mathbb{R}^2\setminus A=\mathbb{R}\times(-\infty,0)\cup\mathbb{R}\times(0,\infty)$$

which is open, so A is closed and thus $A=\overline{A}$. Now for $\operatorname{int}A=\emptyset$. Suppose there ~~was~~ is ~~were~~ some open $U\subseteq\mathbb{R}^2$ with $U\subseteq A$. Then there ~~was~~ is an interval $(a,b)\times(c,d)\subseteq U$. ~~with $((a,b)\times(c,d))\cap\subseteq A$.~~ However, as then ~~$c<0<d$, there are also elements~~ there are also elements $x\times y$ with $y\neq 0$ and $x\times y\in A$. ↯ □

(b) $B=\{x\times y \mid x>0,\ y\neq 0\}$

$\operatorname{int} A=\cancel{\emptyset}\ B$

$\operatorname{bd} A=\cancel{[0,\infty)\times\mathbb{R}}\ \{0\}\times\mathbb{R}\cup(0,\infty)\times\{0\}$

Proof. B is open as $B=(0,\infty)\times(-\infty,0)\cup(0,\infty)\times(0,\infty)$. It is clear that $\overline{B}=[0,\infty)\times\mathbb{R}$. □

(20) (c) $C = A \cup B = (0,\infty) \times \mathbb{R} \cup \mathbb{R} \times \{0\}$

~~$\text{int}\, C = (0,\infty) \times \mathbb{R}$~~

~~$\text{bd}\, C = \mathbb{R} \times \{0\}$~~

~~Proof. C is closed as~~

~~$\mathbb{R}^2 \setminus C = \mathbb{R}^2 \setminus A \cap \mathbb{R}^2 \setminus B$~~

~~$= (-\infty, 0] \times \mathbb{R} \cap (\mathbb{R} \times (-\infty, 0) \cup \mathbb{R} \times (0,\infty))$~~

TODO

(21) ~~(a)~~ SKIPPED

18 Continuous Functions

Definition (Continuity): Let X and Y be topological spaces. A function $f: X \to Y$ is said to be *continuous* if for every open ~~$V \subseteq V$~~, $V \subseteq Y$, the preimage $f^{-1}(V)$ is ~~also~~ open. (Note that if Y is given by a basis $\mathcal{B}$, it suffices to show that the preimage of arbitrary basis elements is open. The same holds if Y is given by a subbasis $\mathcal{S}$.)

Theorem (Equivalent Notions of Continuity): Let X and Y be topological spaces and let $f: X \to Y$. Then the following are equivalent:

(i) f is continuous;

(ii) for every $A \subseteq X$, we have $f(\bar{A}) \subseteq \overline{f(A)}$;

(iii) for every closed $B \subseteq Y$, the set $f^{-1}(B)$ is closed;

(iv) for every $x \in X$ and every neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Definition (Homeomorphism): Let X and Y be topological spaces and let $f: X \to Y$ be a bijection. If both f and its inverse $f^{-1}: Y \to X$ are continuous, f is called a *homeomorphism*. Equivalently, a bijection $f: X \to Y$ is a homeomorphism if $U \subseteq X$ is open iff $f(U)$ is open.

Definition (Topological Property): Any property on X that is entirely expressed in terms of the topology, i.e., in terms of open sets, is called a *topological property*. If Y is a topological space homeomorphic to X, it has the same property.

Definition (Embedding): Let X and Y be topological space and let $f: X \to Y$ be injective. ~~Let $Z = f(X)$~~. Let $Z = f(X)$ be the image of X under f, considered as a subspace of Y. Then the restriction $f': X \to Z$ ~~is~~ of f is bijective. If f' is a homeomorphism of X with Z, the map $f: X \to Y$ is called an *embedding* of X in Y.

Theorem (Construction of Cont. Functions): Let X, Y, and Z be topological spaces. Then:

(a) constant; if $f: X \to Y$ maps all of X to a single point $y_0 \in Y$, then f is continuous;

(b) inclusion; if $A \subseteq X$ is a subspace, the inclusion map $j: A \to X$, $j(a) = a$, is continuous;

(c) composites; if $f: X \to Y$ and $g: Y \to Z$ are continuous, then $g \circ f: X \to Z$ is continuous;

(d) restricting the domain; if $f: X \to Y$ is continuous and $A \subseteq X$ is a subspace, then $f|_A : A \to Y$ is continuous;

(e) restricting the range; if $f: X \to Y$ is continuous and $B \subseteq Y$ is a subspace with $f(X) \subseteq B$, then $g: X \to B$ obtained by restricting the range of f is continuous;

(f) expanding the range; if $B \subseteq Y$ is a subspace and $f: X \to B$ is continuous, then $g: X \to Y$ obtained by expanding the range of f is continuous;

(g) local formulation of continuity; the map $f: X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f|_{U_\alpha} : U_\alpha \to Y$ is continuous for all α.

Theorem (Pasting Lemma): Let X be a topological space and let $A, B \subseteq X$ be closed such that $X = A \cup B$. Let $f: A \to Y$ and $g: B \to Y$ be continuous (where Y is a topological space). If $f(x) = g(x)$ for all $x \in A \cap B$, then $h: X \to Y$ defined as

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B, \end{cases}$$

is continuous. (This also holds if A and B are both open.)

Theorem (Maps into Products): Let $f: A \to X \times Y$ be given by $f(a) = (f_1(a), f_2(a))$. Then f is continuous if and only if both $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous. The maps f_1 and f_2 are called <u>coordinate functions</u> ~~of~~ of f.

Exercises:

(1) Proof. Let $f: \mathbb{R} \to \mathbb{R}$ be continuous according to the ϵ-δ-definition. (That is, for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in B(x_0, \delta)$, $f(x) \in B(f(x_0), \epsilon)$; ~~then~~ for all $x_0 \in \mathbb{R}$.) Let (a,b) be any basis element of the standard topology. ~~Let x_0~~ ~~Let $U = f((a,b))$ be the preimage of (a,b) under~~ Let U be the preimage of (a,b) under f, i.e., $U = f^{-1}((a,b))$. If $U = \emptyset$, it is open. Suppose $U \neq \emptyset$. Let $x \in U$ and choose $\epsilon_x > 0$ such that $(a,b) \supseteq B(f(x), \cancel{\epsilon x)}\ \epsilon_x)$. Then there is a $\delta_x > 0$ ~~so~~ such that for all $\tilde{x} \in B(x, \delta_x)$, ~~so~~ we have $f(\tilde{x}) \in B(f(x), \epsilon_x)$. But then

$$f(B(x,\delta_x)) \subseteq B(f(x), \epsilon_x) \subseteq B(a,b),$$

so $B(x, \delta_x) \subseteq U$ by construction. We now have

$$U = \bigcup_{x \in U} B(x, \delta_x).$$

"$\subseteq$": Let $x \in U$. Then $x \in B(x, \delta_x)$, so $x \in$ RHS. "$\supseteq$" was shown before. Thus, U is ~~the~~ a union of open sets and thus open.

$\square$

~~(2) No. Fix $y_0 \in Y$ and consider $f: X \to Y$, $f(x) = y_0$ for all $x \in X$. Let $A \subseteq X$ and let $x \in X$ be~~

(2) No. Consider $X = Y = \mathbb{R}$ and $f(x) = 0$ for all $x \in X$. Consider $A = [0,1]$. Then every $x \in A$ is a limit point of of A. However, $f(A) = \{0\}$, which does not have any limit points, so $f(x)$ ~~can~~ cannot be one.

(3) (a) Proof. Let $\mathcal{T}$ and $\mathcal{T}'$ be topologies over the set X. We denote the respective topological spaces by X and X'. ~~"$\Leftarrow$": Suppose $\mathcal{T}' \supseteq \mathcal{T}$~~ Let $i: X' \to X$ be the identity function. "$\Leftarrow$": Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Let $U \subseteq X$ be open. Then $i^{-1}(U) = U \subseteq X'$ is also open, so i is continuous. "$\Rightarrow$": ~~So~~ Suppose i is continuous. Let $U \in \mathcal{T}$. Then $i^{-1}(U) = U \in \mathcal{T}'$. Thus, $\mathcal{T}' \supseteq \mathcal{T}$.

$\square$

(b) $\to$ next pg.

(3) (b) Proof. Let X and X' be topological spaces over the same set with topologies $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $i: X' \to X$ be the identity function. "$\Rightarrow$": Suppose i is a homeomorphism. Then i and i^{-1} are continuous and by (a), $\mathcal{T} = \mathcal{T}'$. "$\Leftarrow$": Suppose $\mathcal{T} = \mathcal{T}'$. By (a), i and i^{-1} are continuous (it is clear that i is bijective), so i is a homeomorphism.

$\square$

(4) Proof. Let X and Y be topological spaces. Fix $x_0 \in X$ and $y_0 \in Y$. Define $f: X \to X \times Y$ and $g: Y \to X \times Y$ by

$$f(x) = x \times y_0 \quad \text{and} \quad g(y) = x_0 \times y.$$

We only show ~~that~~ that f is an embedding as the case for g is analogous. Clearly, f is injective. Thus, ~~as~~ with $f(X) = X \times \{y_0\}$, $f': X \to X \times \{y_0\}$ is bijective (where f' is just the restriction of f). We can write $f'(x) = f_1(x) \times f_2(x)$ with coordinates $f_1: X \to X: x \mapsto x$ and $f_2: X \to \{y_0\}: x \mapsto y_0$. These are continuous, so f' is. Define $(f')^{-1}: X \times \{y_0\} \to X$ by $(f')^{-1}(x, y_0) = x$. ~~We~~ We show that $f' \circ (f')^{-1} = i$ and $(f')^{-1} \circ f' = i$,

$$(f' \circ (f')^{-1})(x, y_0) = f'((f')^{-1}(x, y_0)) = f'(x) = (x, y_0);$$

$$((f')^{-1} \circ f')(x) = (f')^{-1}(f'(x)) = (f')^{-1}(x, y_0) = x.$$

Let $U \subseteq X$ be open. ~~Then $(f')^{-1}(U) = U \times \{y_0\}$~~ Then $((f')^{-1})^{-1}(U) = U \times \{y_0\}$, which is open in $X \times \{y_0\}$. Thus, f' is a homeomorphism, so f is an embedding.

$\square$

(5) ~~(a)~~ Proof. Let $a, b \in \mathbb{R}$, $a < b$. Consider $f: (0,1) \to (a,b)$ with $f(x) = (b-a)x + a$. Clearly, f is bijective and continuous, so (a,b) and $(0,1)$ are homeomorphic. Now consider $f: [0,1] \to [a,b]$ defined equivalently. The same argument ("trivial") holds.

$\square$

(6) TODO

(7) (a) Proof. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous from the right, i.e.,

$$\lim_{x\searrow a} f(x) = f(a) \quad \text{for all } a\in\mathbb{R}.$$

Consider f ~~from~~ as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$. Let ~~(a,b)~~ $a,b\in\mathbb{R}$, $a<b$. Consider (a,b), a basis element of $\mathbb{R}$. Let $U=f^{-1}((a,b))$. We need to show that U is ~~an open~~ open in $\mathbb{R}_\ell$, i.e., that

$$U=\bigcup_\alpha [a_\alpha,b_\alpha),$$

~~for a collection $\{[a_\alpha,b_\alpha)\}$ of basis elements of $\mathbb{R}_\ell$. "$\supseteq$": Let $x\in[a_\alpha,b_\alpha)$ for some interval.~~
where $[a_\alpha,b_\alpha)$ are all basis elements such that

$$f([a_\alpha,b_\alpha))\subseteq(a,b).$$

"$\supseteq$" is clear (as $f([a_\alpha,b_\alpha))\subseteq(a,b)$, $[a_\alpha,b_\alpha)\subseteq U$).
"$\subseteq$": Let $x\in U$. Then $f(x)\in(a,b)$. ~~Let $(x_n)\subseteq\mathbb{R}$, $x_n\searrow x$, be any sequence such that $x_n\to x$. Then also $f(x_n)\to f(x)$ by assumption. Let $\epsilon>0$ such that $(x-\epsilon,x+\epsilon)\subseteq(a,b)$.~~ Let $\epsilon>0$ such that ~~$(x-\epsilon,x+\epsilon)\subseteq(a,b)$. Then there is~~ $(f(x)-\epsilon,f(x)+\epsilon)\subseteq(a,b)$. Then there is a $\delta>0$ such that for all $\tilde{x}\in(x,x+\delta)$, $f(\tilde{x})\in(f(x)-\epsilon,f(x)+\epsilon)$. That is, $f((x,x+\delta))\subseteq(f(x)-\epsilon,f(x)+\epsilon)\subseteq(a,b)$. Moreover, as ~~$f(x)=$~~ $f(x)\in(a,b)$, so $f([x,x+\delta))\subseteq(a,b)$. Thus, $x\in$ RHS as $[x,x+\delta)$ is a basis element. This shows that f is continuous as a function from $\mathbb{R}_\ell$ to $\mathbb{R}$. $\square$

(b) $f:\mathbb{R}\to\mathbb{R}_\ell$ no functions

$f:\mathbb{R}_\ell\to\mathbb{R}_\ell$ all "usually continuous" functions

$\to$ Abd in chapter 3

~~(8) (a) Proof. Let X be a topological space and let Y be an ordered set with order topology. Let $f,g:X\to Y$ be continuous. Consider $C=\{x\in X \mid f(x)\leq g(x)\}$. We show $\bar{C}=C$. Let $x\in\bar{C}$ and let $a,b\in Y$ with $x\in(a,b)$ Let $x\in\bar{C}$ and a let $U\subseteq X$ be open with $x\in U$.~~

~~(8) (a) Proof. Set $C = \{x \in X \mid f(x) \leq g(x)\}$. Let $x \in \bar{C}$. Then for all open $U \subseteq X$ with $x \in U$, $C \cap U \neq \emptyset$. Suppose $x \notin C$. $x \notin C$. That is, $f(x) > g(x)$.~~

(8) (a) ~~TODO see below~~ TODO

(b) Proof. Let X be a topological space. Let Y be ordered with the ~~order~~ order topology, let $f, g : X \to Y$ be continuous, and define $h : X \to Y$ by

$$h(x) = \min\{f(x), g(x)\}.$$

Set $C = \{x \in X \mid f(x) \leq g(x)\}$ and $C' = \{x \in X \mid g(x) \leq f(x)\}$. By symmetry, both of these sets ~~a~~ are closed and we ~~can~~ can write h as

$$h(x) = \begin{cases} f(x) & \text{if } x \in C, \\ g(x) & \text{if } x \in C'. \end{cases}$$

By the pasting lemma, h is continuous. $\square$

~~(5) Let X be a topological space and let $\{A_\alpha\}$ be a collection of subsets of X such that $X = \bigcup_\alpha A_\alpha$. Let $f : X \to Y$ be a function to a topological space Y such that $f|_{A_\alpha}$ is continuous for each α.~~

~~(a) Proof. Suppose $\{A_\alpha\}$ is finite and each A_α is closed.~~

~~(a) Proof. Let X be a topological space, and let Y be an ordered set with order topology. Let f. $f, g : X \to Y$ be continuous and let~~

$$\cancel{C = \{x \in X \mid f(x) \leq g(x)\}}.$$

~~We show that $X \setminus C$ is open, i.e., C is closed. We have~~

$$\cancel{X \setminus C = \{x \in X \mid f(x) > g(x)\}}$$
$$\cancel{= \bigcup_{\tilde{x} \in X} \{x \in X \mid f(x) > g(\tilde{x})\}}$$

(9) TODO

(10) Proof. Let A, B, C, D be topological spaces and let $f: A \to B$ and $g: C \to D$ be continuous. Define $f \times g: A \times C \to B \times D$ by

$$(f \times g)(a \times c) = f(a) \times g(c) \quad \text{for all } a \times c \in A \times C.$$

Let $U \subseteq B$ and $V \subseteq D$ be open. Then $U \times V$ is any basis element of $B \times D$. We then have

$$(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V), \tag{*}$$

where $f^{-1}(U)$ and $g^{-1}(V)$ are open in A and C, respectively. Thus, $(f \times g)^{-1}(U \times V)$ is open in $A \times C$, so $f \times g$ is continuous. ~~We~~

We need to show that (*) actually holds. "$\subseteq$": Let $a \times c \in (f \times g)^{-1}(U \times V)$. Then $f(a) \in U$ and $g(c) \in V$, so $a \times c \in f^{-1}(U) \times g^{-1}(V)$. "$\supseteq$": Let $a \times c \in f^{-1}(U) \times g^{-1}(V)$. Then $f(a) \in U$ and $g(c) \in V$, so $a \times c \in (f \times g)^{-1}(U \times V)$.

$\square$

(11) Proof. Let $F: X \times Y \to Z$ be continuous. ~~Let~~ Fix $y_0 \in Y$ and define $h: X \to Z$ by $h(x) = F(x \times y_0)$ for all $x \in X$. We can ~~make~~ write $h = F \circ \tilde{h}$ with $\tilde{h}: X \to X \times Y$ defined by $\tilde{h}(x) = x \times y_0$. Clearly, $\tilde{h}$ is continuous and thus h is the composition of continuous functions and therefore itself continuous. The ~~same~~ same ~~bol~~ holds for $k: Y \to Z$, $y \mapsto F(x_0 \times y)$ and fixed $x_0 \in X$. ~~Thus~~ Hence, F is continuous in each variable separately.

$\square$

~~(12)~~

(12) Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } x \times y \neq 0 \times 0, \\ 0 & \text{if } x \times y = 0 \times 0. \end{cases}$$

(a) Proof. We first show that F is continuous in the first argument. ~~Let $h : \mathbb{R} \to \mathbb{R}$ be defi~~ Fix $y_0 \in \mathbb{R}$ and define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = F(x, y_0)$. If $y_0 = 0$, we ~~h~~ have $h(x) = 0$ for all $x \in \mathbb{R}$, ~~so~~ so h is trivially continuous. Suppose $y_0 \neq 0$. Then

$$h(x) = x y_0 / (x^2 + y_0^2).$$

Clearly, h is ~~cont~~ continuous as $x^2 + y_0^2 \neq 0$ for all $x \in \mathbb{R}$. ~~Thus~~ The case for the second argument is completely ~~ang~~ analogous. ~~Thu~~ ~~The~~ Thus, F is continuous in each variable separately. $\square$

(b)
$$g(x) = F(x, x) = \begin{cases} x^2/(2x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases} = \begin{cases} 1/2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

(c) Proof. Suppose F is ~~contonu~~ continuous. But then for all nonempty $A \subseteq X \times \cancel{X} Y$ $F|_A$ is continuous. Let $A = \{x \times x \mid x \in X\}$. Then g corresponds to $F|_A$. ↯ g is clearly not continuous, so F ~~con~~ cannot be continuous. ∎

(13) Proof. Let X, Y be topological spaces. Let Y be Hausdorff, let ~~f~~ $A \subseteq X$, and let $f : A \to Y$ be continuous. Suppose that f can be extended to a continuous $g : \bar{A} \to Y$, i.e., $f(x) = g(x)$ for all $x \in A$. We show that g is unique. Suppose $g' : \bar{A} \to Y$ is another extension. Let $x \in \bar{A}$. If $x \in A$, then $f(x) = g(x) = g'(x)$. Suppose $x \notin A$. Moreover, suppose, for contradiction, that $g(x) \neq g'(x)$. Let $V, V' \subseteq Y$ be open such that $g(x) \in V$ and $g'(x) \in V'$ with $V \cap V' = \emptyset$. (As Y is Hausdorff, such open sets exist.) Then $U = g^{-1}(V)$ and $U' = (g')^{-1}(V')$ ~~are~~ are open with $x \in U$ and $x \in U'$. ~~Thus, $A \cap U \neq \emptyset$ and $A \cap U' \neq \emptyset$.~~ Moreover, $U \cap U'$ is open and ~~$A \cap (U \cap U') \neq \emptyset$~~ $x \in U \cap U'$, so $A \cap (U \cap U') \neq \emptyset$. Let $x_0 \in A \cap (U \cap U')$. Then $f(x_0) = g(x_0) = g'(x_0)$. ↯ This implies that V and ~~V~~ V' are not disjoint. Hence, ~~g~~ $g = g'$ and g is uniquely determined by f. $\square$

19 <u>The Product Topology</u>

Definition (Box Topology): Let $\{X_\alpha\}_{\alpha\in J}$ be an ~~indeed~~ indexed family of topological spaces. We call the topology* ~~with basis~~ all sets of the form $\prod_{\alpha\in J} U_\alpha$ where U_α is open in X_α for the product space $\prod_{\alpha\in J} X_\alpha$ the <u>box topology</u>.

*generated by

Definition (Product Topology): Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces and define

$$S_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta \subseteq X_\beta \text{ open}\},$$

where ~~π_β~~ $\pi_\beta : \prod_{\alpha\in J} X_\alpha \to X_\beta$ is the ~~a~~ projection mapping on β. ~~Let~~ Let S denote

$$S = \bigcup_{\beta\in J} S_\beta$$

the union of all collections S_β. The topology generated by the subbasis ~~S~~ S is the <u>product topology</u> and we consider it as the standard topology on $\prod_{\alpha\in J} X_\alpha$.

~~Theorem (Comparison of Box and Product Topology): The box topology on $\prod_{\alpha\in J} X_\alpha$ has as basis elements all sets of the form $\prod U_\alpha$, where $U_\alpha \subseteq X_\alpha$ is open. The pro product topo to~~

Theorem (Comparison of Box and Product Topology):

- The box topology on $\prod X_\alpha$ has as basis elements all sets of the form $\prod U_\alpha$, where $U_\alpha \subseteq X_\alpha$ is open.

- The product topology on $\prod X_\alpha$ has as basis elements all sets of the form $\prod U_\alpha$, where $U_\alpha \subseteq X_\alpha$ is open and $U_\alpha = X_\alpha$ for all but finitely many α's.

Theorem: Let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of topological spaces, each given by a basis $\mathcal{B}_\alpha$. The collection of all sets of the form $\prod B_\alpha$, $B_\alpha \in \mathcal{B}_\alpha$, is a basis for the box topology. The collection of ~~sets~~ all sets of the form $\prod B_\alpha$, $B_\alpha \in \mathcal{B}_\alpha$, with $B_\alpha = X_\alpha$ for all but finitely many α's, is a basis for the product topology.

Theorem: Let $A_\alpha \subseteq X_\alpha$, $\alpha \in J$, be a subspace. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ if both products are given in either the box or the product topology.

Theorem: If each X_α is Hausdorff, then $\prod X_\alpha$ is Hausdorff in both box and product topology.

Theorem (Closure in Box/Product Topology): Let $\{X_\alpha\}$ be an indexed family of spaces, let $A_\alpha \subseteq X_\alpha$, and suppose $\prod X_\alpha$ is given as either the box or product topology. Then:

$$\cancel{\prod \overline{A_\alpha}} \quad \prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

Theorem (Continuity to Product Topology): Let $\{X_\alpha\}_{\alpha \in J}$ be a family of spaces, let $\prod X_\alpha$ ~~be given~~ have the ~~topo~~ product topology, and let $f : A \to \prod X_\alpha$ be given by

$$f(a) = (f_\alpha(a))_{\alpha \in J}, \quad f_\alpha : A \to X_\alpha,$$

where A is a topological space. Then f is continuous if and only if each f_α is continuous.

Exercises:

(1) Repetitive.

(2) Repetitive.

(3) Repetitive.

(4) Proof. Let $X_1, \dots, X_n$ be topological spaces and consider the spaces $(X_1 \times \cdots X_{n-1}) \times X_n$ and $X_1 \times \cdots \times X_n$. Consider the function $i : (X_1 \times \cdots \times X_{n-1}) \times X_n \to X_1 \times \cdots \times X_n$ ~~so~~ given by

$$i((x_1, \dots, x_{n-1}), x_n) = (x_1, \dots, x_n).$$

Clearly, i is ~~invertible~~ bijective and both i and i^{-1} are continuous. Thus, i is a homeomorphism. $\square$

(5) "If f is continuous, each f_α is continuous." also holds for the box topology.

(6) Proof. Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed set of spaces and let $x_1, x_2, \dots$ be a ~~s~~ sequence in the ~~pro~~ product space $\cancel{\prod_{\alpha \in J} X_\alpha}$ $\prod X_\alpha$. "Only if:" Suppose $x_n \to x$ for some $x \in \prod X_\alpha$. ~~Fix~~ Fix $\alpha \in J$. Let $U_\alpha \subseteq X_\alpha$ be open such that ~~x~~ $x(\alpha) \in U_\alpha$. Then $x_n \in \pi_\alpha^{-1}(U_\alpha)$ for almost all n. Thus, $x_n(\alpha) \in U_\alpha$ for almost all n, so $x_n(\alpha) \to x(\alpha)$. "If:" Suppose that $x_n(\alpha) \to x(\alpha)$ for all $\alpha \in J$ and some $x \in \prod X_\alpha$. ~~F~~ Fix x. Let B be a basis element of $\prod X_\alpha$ such that $x \in B$. Then B has the form $B = \prod U_\beta$, $U_\beta \subseteq X_\beta$ open, and $U_\beta = X_\beta$ for almost all β. However, as $x_n(\beta) \to x(\beta)$, we have $x_n(\beta) \in U_\beta$ for almost all n. Hence, $x_n \in \prod U_\beta = B$ for almost all n. ~~He~~ Therefore, $x_n \to x$. $\square$

I feel like it does not work for the box topology, but I also fail to find the flaw in above proof if one uses the box topology.

(7) TODO

(8) *Proof.* Consider sequences $(a_n), (b_n) \in \mathbb{R}^\omega$ where $a_n > 0$ for all n and let $\mathbb{R}^\omega$ be given under the ~~pro~~ product topology. Define $h: \mathbb{R}^\omega \to \mathbb{R}^\omega$ by $(h(x))_n = a_n x_n + b_n$ for all $x \in \mathbb{R}^\omega$ and all n. Clearly, h is bijective with ~~inve~~ inverse $(h^{-1}(y))_n = (y_n - b_n)/a_n$. Denote by $h_{a,b}: \mathbb{R}^\omega \to \mathbb{R}^\omega$ the function ~~$(h(x))_n =$~~ $(h_{a,b}(x))_n = a_n x_n + b_n$. Then, for given a and b, $h(x) = h_{a,b}(x)$ and $h^{-1}(x) = h_{\tilde{a},\tilde{b}}(x)$ with $\tilde{a}_n = 1/a_n$ and $\tilde{b}_n = -b_n/a_n$. To show that h is a homeomorphism, we only need to show that $h_{a,b}$ is continuous for arbitrary $a, b \in \mathbb{R}^\omega$, $a_n > 0$. Let $U \subseteq \mathbb{R}$ be open and let $n \in \mathbb{N}$. Let $\pi_n: \mathbb{R}^\omega \to \mathbb{R}$ be the n^{th} projection. Then $\pi_n^{-1}(U)$ is an ~~a~~ arbitrary subbasis-element of $\mathbb{R}^\omega$. ~~It~~ The open set U is "built" from intervals, so we may write $U = \bigcup_\alpha (a_\alpha, b_\alpha)$. We have

$$h_{a,b}^{-1}(\pi_n^{-1}(U)) = h_{a,b}^{-1}\left(\pi_n^{-1}\left(\bigcup_\alpha (a_\alpha, b_\alpha)\right)\right) = \bigcup_\alpha h_{a,b}^{-1}(\pi_n^{-1}((a_\alpha, b_\alpha))),$$

so if each $h_{a,b}^{-1}(\pi_n^{-1}((a_\alpha, b_\alpha)))$ is open, $h_{a,b}$ is continuous. Let $m \in \mathbb{N}$. If $n = m$, then

$$(h_{a,b}^{-1}(\pi_n^{-1}((a_\alpha, b_\alpha))))_m = ((a_\alpha - b_n)/a_n,\ (b_\alpha - b_n)/a_n),$$

which is clearly open. If ~~n~~ $n \neq m$, then

$$(h_{a,b}^{-1}(\pi_n^{-1}((a_\alpha, b_\alpha))))_m = \mathbb{R},$$

which is also open. Thus, $h_{a,b}$ is continuous and therefore h is a homeomorphism. $\square$

~~Considering $\mathbb{R}^\omega$ under the box topology, h is still a homeomorphism as as~~

If $\mathbb{R}^\omega$ is given under the box topology, h is <u>not</u> a homeomorphism as it is not continuous: Consider the sequences $a, b \in \mathbb{R}^\omega$ with $a_n = 1$ and $b_n = 0$ for all n. Then h reduces to f of example 2 in §19.

(9) Proof. Let $\{A_\alpha\}_{\alpha\in J}$, $J\neq\emptyset$ be an indexed family of nonempty sets. Suppose the axiom of choice holds. Then there is a choice function $c:\{A_\alpha\}\to X$ with $X=\bigcup A_\alpha$ such that, for all $\alpha\in J$, $c(A_\alpha)\in A_\alpha$. But then

$$\prod_{\alpha\in J} c(A_\alpha)\in\prod_{\alpha\in J}A_\alpha,$$

so the cartesian product $\prod A_\alpha$ is nonempty. Conversely, suppose that $\prod A_\alpha$ is nonempty. Then there is a ~~c~~ $c\in\prod A_\alpha$ and we can define $c(A_\alpha)=c_\alpha$ for all $\alpha\in J$, i.e., a choice function. □

~~(10) (a) Proof.~~

(10) Let A be a set, let $\{X_\alpha\}_{\alpha\in J}$ be an indexed family of spaces, and let $\{f_\alpha\}_{\alpha\in J}$ be an indexed family of functions $f_\alpha:A\to X_\alpha$.

~~(a) Proof.~~

(b) Proof. Let $\mathcal{S}_\beta=\{f_\beta^{-1}(U)\mid U\subseteq X_\alpha \text{ open}\}$ and $\mathcal{S}=\bigcup \mathcal{S}_\beta$. We show that under ~~$\mathcal{T}$~~ $\mathcal{T}=\mathcal{T}(\mathcal{S})$, every f_α is continuous. Fix $\alpha\in J$ and let $U\subseteq X_\alpha$ be open. Then $f_\alpha^{-1}(U)\in \mathcal{S}_\beta\subseteq \mathcal{S}\subseteq \mathcal{T}$, so ~~f~~ f_α is continuous. Thus, the topology generated by $\mathcal{S}$ ensures every function f_α is continuous. We need to show that $\mathcal{S}$ is a subbasis. ~~Let a set.~~ Let $a\in A$ ~~and~~ and choose some $\alpha\in J$. Let U be a neighborhood of $f_\alpha(a)$. Then ~~$x\in f_\alpha^{-1}($~~ $a\in f_\alpha^{-1}(U)\in \mathcal{S}_\alpha\subseteq \mathcal{S}$, so $\mathcal{S}$ is indeed a subbasis. $\boxed{*}$

(a) Proof. Let $\mathcal{T}$ be the topology on A generated by $\mathcal{S}$ of (b) and let $\mathcal{T}'$ be another topology on A such that each f_α is continuous and that is the coarsest such topology. We show that $\mathcal{T}=\mathcal{T}'$. As $\mathcal{T}'$ is the ~~coarse~~ coarsest topology, $\mathcal{T}\supseteq\mathcal{T}'$. We need to show $\mathcal{T}\subseteq\mathcal{T}'$. Let $U\in\mathcal{T}$. Then there are ~~$\{B_\beta\}$~~ basis elements $\{B_\beta\}$ (built from finite intersections of elements of $\mathcal{S}$) such that $U=\bigcup B_\beta$. Suppose $B_\beta\in\mathcal{T}'$ for all β, then also ~~U~~ $U\in\mathcal{T}'$. We show that $B_\beta\in\mathcal{T}'$. There are $S^1,S^2,\dots,S^k\in\mathcal{S}$ such that ~~$B_\beta=\bigcap$~~ $B_\beta=S^1\cap S^2\cap\dots\cap S^k$. ~~If~~ Suppose $S^i\in\mathcal{T}'$, then also $B_\beta\in\mathcal{T}'$. We ~~s~~ show that $S^i\in\mathcal{T}'$. Let $\alpha\in J$ ~~such~~ such that $S^i\in\mathcal{S}_\alpha$. Then there is an open $V\subseteq X_\alpha$ such that $S^i=f_\alpha^{-1}(V)$. But then, as f_α is continuous w.r.t. $\mathcal{T}'$, also $S^i\in\mathcal{T}'$. Hence, $\mathcal{T}=\mathcal{T}'$ and the coarsest topology is unique. □

* We need to show $\mathcal{T}$ is coarsest. See next page.

(10) (b) Proof (Cont.): Let $\mathcal{T}'$ be a topology on A ~~sucho~~ such that each f_α is continuous. We show that $\mathcal{T}' \supseteq \mathcal{T}$. This is trivial, cf. (b). □

(c) Proof. Let Y be a topological space and let $g: Y \to A$ be a function. "Only if:" Suppose ~~f~~ g is ~~cont~~ continuous relative to $\mathcal{T}$. Then, as each f_α is continuous, $f_\alpha \circ g$ is continuous. "If:" Suppose that $f_\alpha \circ g$ is continuous for all f_α. We show that g is continuous. Let ~~S be~~ $S \in \mathcal{S}$, then there is some $\alpha \in J$ and an open $U \subseteq X_\alpha$ such that $S = f_\alpha^{-1}(U)$. But then ~~$g^{-1}(U) =$~~

$$\cancel{g^{-1}(U) = g^{-1}(f_\alpha^{-1}(}$$

$$g^{-1}(S) = g^{-1}(f_\alpha^{-1}(U)) = (f_\alpha \circ g)^{-1}(U)$$

is open as $f_\alpha \circ g$ is continuous. Hence, g is continuous. □

~~(d) Proof. Let $f: A \to \prod X_\alpha$ be defined by $f(a) = (f_\alpha(a))_{\alpha \in J}$ for all $a \in A$. Let $Z = f(A)$ be a subspace of $\prod X_\alpha$ under the product topology. Let Let $S \in \mathcal{S}$ be a subbasis element of $\mathcal{T}$. Then there is an $\alpha \in J$ and an open set $U \subseteq X_\alpha$ such that $S = f_\alpha^{-1}(S)$ $S = f_\alpha^{-1}(U)$. We then have $f_\alpha(S) = f_\alpha(f_\alpha^{-1}(U)) = U$~~

(d) TODO

20 <u>The Metric Topology</u>

Definition (Metric): A <u>metric</u> on a set X is a function $d: X \times X \to \mathbb{R}$ such that:

(i) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ iff $x = y$;

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(iii) $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$.

Definition (ϵ-Ball): $B_d(x, \epsilon) = \{ \cancel{x} y \in X \mid d(x,y) < \epsilon \}$.

Definition (Metric Topology): If (X, d) is a metric space, then the collection of all ϵ-balls $B_d(x, \epsilon)$, for $x \in X$ and $\epsilon > 0$, is a basis for a topology on X, the <u>metric topology</u>.

Rephrased: A set $U \subseteq X$ is open in the metric topology on X induced by d if and only if for all $y \in U$ there is an $\epsilon > 0$ such that $B_d(y, \epsilon) \subseteq U$.

Definition (Metrizable): Let X be a topological space. Then X is said to be <u>metrizable</u> if there exists a metric d on X inducing the topology on X. A <u>metric space</u> is a metrizable space X together with a metric d inducing the topology.

Definition (Norm, Euclidean/Square Metric on $\mathbb{R}^n$): Let $x \in \mathbb{R}^n$, $x = (x_1, \dots, x_n)$, we define the <u>norm</u> of x by

$$\|x\| = (x_1^2 + \dots + x_n^2)^{1/2}.$$

We define the <u>Euclidean metric</u> on $\mathbb{R}^n$ by

$$d(x,y) = \|x - y\|.$$

We define the <u>square metric</u> on $\mathbb{R}^n$ by

$$\cancel{\rho(x} \quad \rho(x,y) = \max_{i=1,\dots,n} |x_i - y_i|.$$

Lemma (Finer by Metric): ~~Let~~ Let d, d' be metrics on the set X and let $\mathcal{T}, \mathcal{T}'$ be the induced topologies, respectively. Then $\mathcal{T}'$ is finer than $\mathcal{T}$ if and only if for all $x \in X$ and for all $\epsilon > 0$, there is a $\delta > 0$ such that $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$.

Theorem: The topologies on $\mathbb{R}^n$ induced by the Euclidean metric d and the square metric ρ both equal the product topology.

Definition/Theorem (Uniform Metric): Let J be an index set, given points $x, y \in \mathbb{R}^J$, the <u>uniform metric</u> $\bar{\rho}$ on $\mathbb{R}^J$ is defined by

$$\bar{\rho}(x,y) = \sup\{\bar{d}(x_\alpha, y_\alpha) \mid \alpha \in J\},$$

where $\bar{d}(x_\alpha, y_\alpha) = \min\{|x_\alpha - y_\alpha|, 1\}$. The topology induced by $\bar{\rho}$ is the <u>uniform topology</u>. It is finer than the product topology and coarser than the box topology. If J is infinite, all three are different.

Theorem (Metrizability of $\mathbb{R}^\omega$): Let $\bar{d}(x,y) = \min\{|x-y|, 1\}$ be the standard bounded ~~topology~~ metric on $\mathbb{R}$. If $x, y \in \mathbb{R}^\omega$, define

$$D(x,y) = \sup_i \bar{d}(x_i, y_i)/i.$$

Then D induces the product topology on $\mathbb{R}^\omega$.

Exercises: (I skipped most exercises...)

(1) (a) *Proof.* Consider $\mathbb{R}^n$ and define $d' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d'(x, y) = \sum_{i=1}^n |x_i - y_i|.$$

We first show that d' is a metric. Properties (i) and (ii) ~~a~~ are trivial. For ~~H~~ (iii), let $x, y, z \in \mathbb{R}^n$. Then,

$$d'(x, z) = \sum_i |x_i - z_i| \leq \sum_i (|x_i - y_i| + |y_i - z_i|)$$

$$= \sum_i |x_i - y_i| + \sum_i |y_i - z_i| = d'(x, y) + d'(y, z).$$

~~Th~~ Thus, d' is ~~inded~~ ~~on~~ ~~indeed~~ indeed a metric on $\mathbb{R}^n$. We now show that the topology induced by d' equals the usual topology on $\mathbb{R}^n$. Let $\mathcal{T}$, $\mathcal{T}_\rho$ and $\mathcal{T}'$ be the topologies induced by ~~the~~ ~~Euclidean metric,~~ the ~~sq~~ square metric, and d', respectively. Recall that $\mathcal{T}_\rho = \mathcal{T}$. ~~We show that $\mathcal{T}' \supseteq \mathcal{T}_\rho$. Let $x, y \in \mathbb{R}^n$. Then $d'(x,y) \leq n\,\rho(x,y)$. Let $\epsilon > 0$ and set $\delta = \epsilon/n$. Then $B_{d'}(x, \delta) \subseteq B_\rho(x, \epsilon)$ for all Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Set $\delta = \epsilon/n$. Then~~

$$\cancel{B_{d'}(x, \delta) \subseteq B_\rho(x, \epsilon)}$$

~~as for all $y \in B_{d'}(x, \delta)$ we have $d'(x,y) < \delta = \epsilon/n$, so $d'(x,y)$~~ We show that $\mathcal{T}_\rho \supseteq \mathcal{T}'$. Let $x, y \in \mathbb{R}^n$. Then $d'(x,y) \leq n\,\rho(x,y)$. Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Then

$$B_\rho(x, \epsilon/n) \subseteq B_{d'}(x, \epsilon),$$

as for all $y \in B_\rho(x, \epsilon/n)$, we have

$$d'(x, y) \leq n\,\rho(x, y) < n \cdot \epsilon/n = \epsilon,$$

so $y \in B_{d'}(x, \epsilon)$. Hence, $\mathcal{T}_\rho \supseteq \mathcal{T}'$. Conversely, note that $\rho(x,y) \leq d'(x,y)$ for all $x, y \in \mathbb{R}^n$. Following the above argument, we have $\mathcal{T}' \supseteq \mathcal{T}_\rho$. Hence, $\mathcal{T}' = \mathcal{T}_\rho$, so the topology induced by d' equals the usual topology on $\mathbb{R}^n$. $\square$

For $n = 2$, we ~~ha~~ have basis elements of the form:

(Only the interior.)

(1) (b) All metrics on finite-dim. metric spaces are equivalent.

(2) TODO

~~(3) (a) Proof. Let (X, d) be a metric space.~~

(3) (a) TODO

(b) Proof. Let (X, d) be a metric space and let X' be another space over the same set X. Suppose that $d : X' \times X' \to \mathbb{R}$, i.e., the metric d under the topology of X', is continuous. We show that the topology of X' is finer than the topology of X. ~~Let $x_0 \in X$ and let $\epsilon > 0$. Consider $B(x_0, \epsilon)$. Then, for all $x, y \in B(x_0, \epsilon)$, we have~~ ~~have $d(x, y) < 2\epsilon$:~~

$$\cancel{d(x,y) \leq d(x, x_0) + d(x_0, y) < 2\epsilon}$$

~~Hence, $d(B(x_0, \epsilon) \times B(x_0, \epsilon))$~~ Hence,

$$\cancel{B(x_0, \epsilon) \times B(x_0, \epsilon) \subseteq d^{-1}((-2\epsilon, 2\epsilon))}.$$

~~(Note that d is nonnegative, so the interval $(-2\epsilon, 2\epsilon)$ is a bit "too large," but makes it open and symmetric.)~~ Let $U \subseteq X$ be open. Then for all $x \in U$ there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. We show ~~that~~ that U is also open w.r.t. X'.

TODO

(4) (a) $f(t) = (t, 2t, 3t, \ldots)$

Product topology: Continuous. Let $B = \prod_i B_{\alpha_i}$ be a basis element, where $B_{\alpha_i} = \mathbb{R}$ for almost all α_i, $i \in \mathbb{N}$. For B_i, $i \in \mathbb{N}$ with $B_i \neq \mathbb{R}$, we have $B_i = (x_i, y_i)$, $x_i < y_i$. We thus have

$$f^{-1}(B) = \bigcap_{i=1}^{\infty} \frac{1}{i} B_i \overset{\Delta}{=} \bigcap_{i=1}^{N} (x_i/i, y_i/i),$$

supposing only for $1 \leq i \leq N$ we have $B_i \neq \mathbb{R}$. As the RHS is a finite intersection ~~of~~ of open sets, it is itself open. □

Uniform topology: TODO

Box topology: Not continuous. Consider the basis element ~~$\in$~~ $B = \prod_{n=1}^{\infty} (-1/\cancel{n}, 1/\cancel{n})$. Then $f^{-1}(B) = \{0\}$, which is not open. Suppose there where a $x \in f^{-1}(B)$, $x \neq 0$. Then there is some $n \in \mathbb{N}$ such that $1/n < |x|$. Consequently, ~~we~~ $n|x| > 1/\cancel{n}$, so $nx \notin (-1/\cancel{n}, 1/\cancel{n})$. Hence, $f(x) \notin B$. □

(4) (a) $g(t)=(t,t,t,\dots)$

Product topology: Continuous as each coordinate is continuous.

Uniform topology: TODO

Box topology: Consider $B=\prod_{n=1}^{\infty}(-1/n,\ 1/n)$ and follow the same argument as for g.

$h(t)=(t,\ t/2,\ t/3,\ \dots)$

Product topology: Continuous.

~~Un~~ Uniform topology: TODO

Box topology: Not continuous, consider $B=\prod_{n=1}^{\infty}(-\frac{1}{n^2},\frac{1}{n^2})$.

(b) TODO

(5) TODO

(6) TODO

(7) TODO

(8) TODO

~~(9) (a) Proof. Let $x,y,z\in\mathbb{R}^n$. Then, using Einstein summation,~~

$$\cancel{x\cdot(y+z)=x_i(y+z)_i=x_i(y_i+z_i)=x_iy_i+x_iz_i=x\cdot y+x\cdot z}$$

□

~~(b) Proof. Let $x,y\in\mathbb{R}^n$ and suppose $x,y\neq 0$. (If $x=0$ or $y=0$, things are trivial.)~~

~~(b) Proof. Let $x,y\in\mathbb{R}^n$. Then~~

$$\cancel{0\le\|x-y\|}$$

(9) TODO

(10) TODO

(11) TODO

The Metric Topology (continued)

Proposition (Basic Properties):

- Subspaces are well-behaved: If A is a subspace of ~~an a metric space~~ topological space X and d is a metric on X, then $d|_{A\times A}$ is a metric for the topology of A.

- The Hausdorff axiom is satisfied.

- All countable products of metric spaces are metrizable.

-

Theorem (Continuity in Metric Spaces): Let $f: X \to Y$; let X and Y be metrizable with metrics d_X and d_Y, respectively. Then f is continuous if and only ~~if for~~ if: for all $x \in X$ and all $\epsilon > 0$, there is a $\delta > 0$ such that for all $x' \in X$ with $d_X(x, x') < \delta$, ~~also~~ we have $d_Y(f(x), f(x')) < \epsilon$.

Lemma (Sequence Lemma): Let X be a topological space and let $A \subseteq X$. Let $x \in X$. If there is a sequence $(x_n) \subseteq A$ with $x_n \to x$, then $x \in \bar{A}$. The converse holds if X is metrizable.

Theorem (Sequence Theorem): Let $f: X \to Y$. If f is continuous, then for every convergent sequence $x_n \to x$ in X, we have $f(x_n) \to f(x)$. The converse holds if X is metrizable.

Lemma: Addition, subtraction, and multiplication are continuous functions from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$; and ~~the~~ division is a continuous function from $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$ into $\mathbb{R}$. (If $\mathbb{R}$ is given under a metrizable topology.)

Definition (Uniform Convergence): Let $f_n: X \to Y$ be a sequence of functions from the set X to the metric space (Y, d). Then (f_n) converges uniformly to a function $f: X \to Y$ if for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \epsilon$ for all $x \in X$ and all $n \geq N$.

Theorem (Uniform Limit): Let $f_n: X \to Y$ be a sequence of functions from a topological space X to a metric space Y. If $(f_n) \to f$ uniformly for some $f: X \to Y$, and every f_n is continuous, then f is continuous.

Exercises:

(1) Proof. Let (X, d) be a metric space and let $A \subseteq X$ be a subspace. We ~~show~~ show that the topology A inherits from X equals the topology induced by $d|_{A \times A}$ over A. Let

$$\mathcal{B} = \{ B(x, \epsilon) \cap A \mid x \in X, \ \epsilon > 0 \} \quad \text{and}$$

$$\mathcal{B}' = \{ B_A(x', \epsilon) \mid x' \in A, \ \epsilon > 0 \}$$

be the ~~span~~ subspace and metric basis, respectively. We ~~fox~~ first show that $\mathcal{T}(\mathcal{B}')$ is finer than $\mathcal{T}(\mathcal{B})$. ~~Let $B' \in \mathcal{B}'$, $B' = B_A(x, \epsilon)$ B~~ Let $B_A(x', \epsilon) \in \mathcal{B}'$ and let $x \in B_A(x', \epsilon)$. Select a δ-ball $B_A(x, \delta) \subseteq B_A(x', \epsilon)$. As $x \in A$, we have $x \in B(x, \delta) \cap A \subseteq B_A(x, \delta)$. Hence, $\mathcal{T}(\mathcal{B}')$ is finer than $\mathcal{T}(\mathcal{B})$. To show the converse, consider $B(x', \epsilon) \cap A \in \mathcal{B}$. Let $x \in B(x', \epsilon) \cap A$. ~~As the latter is open~~ Then there is a $\delta > 0$ such that $B(x, \delta) \subseteq B(x', \epsilon)$. Consequently, $B(x, \delta) \cap A \subseteq B(x', \epsilon) \cap A$ with $x \in A$. But then

$$\begin{aligned} B(x, \delta) \cap A &= \{ y \in X \mid d(x, y) < \delta \} \cap A \\ &= \{ y \in A \mid d(x, y) < \delta \} \\ &= \{ y \in A \mid d|_{A \times A}(x, y) < \delta \} \\ &= B_A(x, \delta) \in \mathcal{B}'. \end{aligned}$$

Hence, $\mathcal{T}(\mathcal{B})$ is finer than $\mathcal{T}(\mathcal{B}')$, so $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$. □

(2) Proof. Let (X, d_X) and (Y, d_Y) be metric spaces and let $f: X \to Y$ be an isometry, i.e., $d_X(x, x') = d_Y(f(x), f(x'))$ for all $x, x' \in X$. Clearly, f is injective: ~~but~~ let $x, x' \in X$, $x \neq x'$, and suppose $f(x) = f(x')$. ~~But~~ But then $0 < d_X(x, x') = d_Y(f(x), f(x')) = 0$. ↯ Let $A = f(X)$ and define $g: X \to A$, $g(x) = f(x)$. g is bijective. To show that g is continuous, let $y \in A$ and $\epsilon > 0$. Set* $U = g^{-1}(B_Y(y, \epsilon))$ and let $x \in U$. Then $f(x) \in B_Y(y, \epsilon)$, so there is a δ-ball with $B_Y(f(x), \delta) \subseteq B_Y(y, \epsilon)$. We show that $B_X(x, \delta) \subseteq U$. Let $x' \in B_X(x, \delta)$. Then

$$\delta > d_X(x, x') = d_Y(f(x), f(x')),$$

so $f(x') \in B_Y(f(x), \delta) \subseteq B_Y(y, \epsilon)$. Hence, $x' \in U$ and as x was arbitrary, U is open. Thus, g is continuous and as g^{-1} is 'symmetric', it is continuous, too. Therefore, f is an embedding. □

* Assume $f = g$ appropriately (notation typo).

(3) (a) Proof. Let $(X_1, d_1), \dots, (X_n, d_n)$ be metric spaces and let

~~$\rho : X_1 \times \dots \times X_n \to \mathbb{R}$~~

$$\rho(x, y) = \max_{i=1}^{n} d_i(x_i, y_i).$$

Clearly, ρ is a metric over $X_1 \times \dots \times X_n$. We show that the product topology $\mathcal{T}$ equals the metric topology $\mathcal{T}_\rho$. Let

$$\mathcal{B} = \{ B_1(x_1, \epsilon_1) \times \dots \times B_n(x_n, \epsilon_n) \mid \dots \} \quad \text{and}$$

$$\mathcal{B}_\rho = \{ B_\rho(x, \epsilon) \mid \dots \}$$

be the bases, respectively. "$\mathcal{T} \supseteq \mathcal{T}_\rho$": Let $B_\rho(x, \epsilon) \in \mathcal{B}_\rho$ and let $x' \in B_\rho(x, \epsilon)$. Then there is a $\delta > 0$ such that $B_\rho(x', \delta) \subseteq B_\rho(x, \epsilon)$. But then

$$\mathcal{B} \ni B_1(x', \delta) \times \dots \times B_n(x', \delta) \subseteq B_\rho(x', \delta),$$

so $\mathcal{T}$ is finer ~~than~~ than $\mathcal{T}_\rho$. "$\mathcal{T}_\rho \supseteq \mathcal{T}$": Let $B_1(x_1, \epsilon_1) \times \dots \times B_n(x_n, \epsilon_n) \in \mathcal{B}$. ~~Then there is an $x \in X_1 \times \dots \times X_n$ with $x \in$~~ ~~Then there is an~~ ~~Then~~ Let $x' \in B_1(x_1, \epsilon_1) \times \dots \times B_n(x_n, \epsilon_n)$ and let $\delta_1, \dots, \delta_n > 0$ such that

$$B_1(x_1', \delta_1) \times \dots \times B_n(x_n', \delta_n) \subseteq B_1(x_1, \epsilon_1) \times \dots \times B_n(x_n, \epsilon_n).$$

Set $\delta = \min_{i=1}^{n} \delta_i$. Then

$$B_\rho(x', \delta) \subseteq B_1(x_1', \delta_1) \times \dots \times B_n(x_n', \delta_n),$$

so $\mathcal{T}_\rho$ is finer than $\mathcal{T}$ and thus, $\mathcal{T} = \mathcal{T}_\rho$.

$\square$

~~(b) Proof. Let, for all $n \in \mathbb{N}$, (X_n, d_n) be a metric space~~

(b) Proof. Let, for all $i \in \mathbb{N}$, (X_i, d_i) be a metric space. Let $\bar{d}_i = \min\{d_i, 1\}$ and define

$$D(x, y) = \sup_i (\bar{d}_i(x_i, y_i)/i)$$

over $\prod X_i$. Clearly, D is a metric. Denote by $\mathcal{B}$ and $\mathcal{B}'$ the product ~~basis~~ and metric bases, ~~$\mathcal{B} = \{$~~ respectively (i.e., $\mathcal{T}(\mathcal{B})$ is the product topology on $\prod X_i$ and $\mathcal{T}(\mathcal{B}')$ is the ~~rat~~ metric topology). We show that $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$. "$\mathcal{T}(\mathcal{B}') \supseteq \mathcal{T}(\mathcal{B})$": Let ~~$B' \in \mathcal{B}'$~~ $B \in \mathcal{B}$, then B has the form $B = \prod B_i$ with, say, $B_i = X_i$ for all $i > n$. For $i \leq n$, B_i is some ball, $B_i = B_{d_i}(x_i, \epsilon_i)$. Let $x' \in B$, then there are $\delta_i > 0$ such that $B_{d_i}(x_i', \delta_i) \subseteq B_i$ for all $i \leq n$. Select each δ_i such that $\delta_i \leq 1$. Then

→ next page

Let $\delta = \min\{\delta_i / i \mid i = 1, \dots, n\}$. Then

$$B_D(x', \delta) \subseteq B.$$

To see this, let $y \in B_D(x', \delta)$. Consider y_i. If $i \le n$, then

$$\bar{d}_i(x'_i, y_i)/i \le D(x', y) < \delta \le \delta_i / i,$$

so $\bar{d}_i(x'_i, y_i) < \delta_i \le 1$. Hence, also $d_i(x'_i, y_i) < \delta_i$, and we have ~~$y_i \in B_i$~~ $y_i \in B_{d_i}(x'_i, \delta_i) = B_i$. If $i > n$, then $B_i = X_i$, so $y_i \in B_i$ is trivial. Hence, $B_D(x', \delta) \subseteq B$, so $\mathcal{T}(\mathcal{B}')$ is finer than $\mathcal{T}(\mathcal{B})$. "$\mathcal{T}(\mathcal{B}) \supseteq \mathcal{T}(\mathcal{B}')$": Let $U \in \mathcal{T}(\mathcal{B})$ and let $x \in U$. Then there is an $\epsilon > 0$ such that $B_D(x, \epsilon) \subseteq U$. Choose $N \in \mathbb{N}$ such that $1/N < \epsilon$. Set

$$V = B_{d_1}(x_1, \epsilon) \times \dots \times B_{d_N}(x_N, \epsilon) \times X_{N+1} \times \dots .$$

We show that $V \subseteq B_D(x, \epsilon)$. (Note that $x \in V$ is trivial.) ~~Let $i \in \mathbb{N}$. If $i > N$~~ For all $y \in \prod X_i$ and all $i \ge N$, we have $\bar{d}_i(x, y)/i \le 1/N$. Hence, for all $y \in V$, we have

$$\begin{aligned} D(x, y) &= \sup_i (\bar{d}_i(x_i, y_i)/i) \\ &= \max\left\{ \frac{\bar{d}_1(x_1, y_1)}{1}, \dots, \frac{\bar{d}_N(x_N, y_N)}{N}, \frac{1}{N} \right\} \end{aligned}$$

However, for $i \le N$, we have $\bar{d}_i(x_i, y_i) < \epsilon \le 1$, so $\bar{d}_i(x_i, y_i)/i < \epsilon / i$. Therefore,

$$D(x, y) = \max\{\epsilon/1, \epsilon/2, \dots, \epsilon/N, 1/N\} < \epsilon,$$

so $y \in B_D(x, \epsilon)$. Thus, $\mathcal{T}(\mathcal{B})$ is finer than $\mathcal{T}(\mathcal{B}')$ and we get $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$.

$\square$

(4) Deferred to ch. 4.

(5) Proof. Let X ~~and Y~~ be a metrizable space and let $\circ : X \times X \to X$ be a binary continuous operation. ~~Let $X \times X$ be equ~~ Let $(x_n), (y_n) \subseteq X$ be sequences converging to x, y, respectively. Then $x_n \times y_n \to x \times y$. As $\circ$ is continuous, $x_n \circ y_n \to x \circ y$.

Let $X = \mathbb{R}$. Then for $\circ$ as addition, subtraction, and multiplication, the above implies that $x_n + y_n \to x + y$, $x_n - y_n \to x - y$, and $x_n y_n \to xy$ for $x_n \to x$ and $y_n \to y$.

Showing this for division is analogous.

$\square$

(6) Proof. Let $f_n : [0,1] \to \mathbb{R}$ be given by $f_n(x) = x^n$ for all $n \in \mathbb{N}$. ~~A~~ Fix $x \in [0,1]$. If $x = 1$, then $f_n(x) = 1$ for all n, so $(f_n(x)) \to 1$ is trivial. Suppose $x < 1$. We show that $(f_n(x)) \to 0$. Let $U \subseteq \mathbb{R}$ be open with $0 \in U$. Then there is a ball $B(0, \epsilon) \subseteq U$. Choose $N \in \mathbb{N}$ such that $x^n < \epsilon$ for all $n \geq N$. Then $(f_n(x)) \to 0$. We can always find such an N as $x < 1$.

However, f_n does not converge uniformly. Suppose it does. ~~Then~~ Then the pointwise limit equals the uniform limit. ~~i.e., $f_n \to f$ and $f_n \to f$ uniformly~~ ~~and~~ From before, we have the pointwise limit

$$f : [0,1] \to \mathbb{R} : x \mapsto \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x = 1. \end{cases}$$

However, as each f_n is continuous, f would need to be continuous. ↯ Hence, f_n does not converge uniformly.

(7) Proof. Let X be a set, let $f_n : X \to \mathbb{R}$ be a sequence of functions, and let $\bar{\rho}$ be the ~~unifom~~ uniform metric over $\mathbb{R}^X$. Let $f : X \to \mathbb{R}$ be ~~an~~ a function. "If:" Suppose (f_n) converges to f in the metric space $(\mathbb{R}^X, \bar{\rho})$. Let $\epsilon > 0$. then $f_n \in B_{\bar{\rho}}(f, \epsilon)$ ~~for almost all~~ eventually. ~~Let $x \in X$. Suppose $\epsilon \leq 1$. Let $x \in X$. Then~~ Suppose $\epsilon < 1$. Then $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$ by definition of the ~~$|f_n(x) - f(x)|$~~ uniform metric $\bar{\rho}$. Hence, $f_n \to f$ uniformly. "Only if:" Suppose $f_n \to f$ uniformly. Let $\epsilon > 0$. Then $|f_n(x) - f(x)| < \epsilon$ holds for all $x \in X$ eventually. Hence, we have

$$\bar{\rho}(f_n, f) = \sup_{x \in X} \min\{|f_n(x) - f(x)|, 1\} < \epsilon$$

$$\cancel{\leq \sup_{x \in X} \min\{\epsilon, 1\} = \epsilon}$$

if we choose $\epsilon \leq 1$. Thus, $f_n \to f$ in $\mathbb{R}^X$.

(8) Proof. Let X be a topological space and let Y be a metric space. Let $f_n : X \to Y$ be a sequence of continuous functions converging ~~on~~ uniformly to $f : X \to Y$. Let $(x_n) \subseteq X$, $x_n \to x \in X$. We show that, in Y, we have $f_n(x_n) \to f(x)$ as $n \to \infty$. ~~Let $\epsilon > 0$. Recall that as $x_n \to x$, we have $f_n(x_n) \to f_n f$~~ We thus have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)|$$

$$\leq \underbrace{\left(\sup_x |f_n(x) - f(x)|\right)}_{\to 0} + \underbrace{|f(x_n) - f(x)|}_{\to 0} \to 0,$$

so $f_n(x_n) \to f(x)$.

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(9) Repetitive (from analysis).

(10) For $A = \{x \times y \mid xy = 1\}$ we ~~can~~ can write

$$f : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto xy; \qquad A = f^{-1}(\{1\}).$$

As $\{1\}$ is closed in $\mathbb{R}$, A is closed (as f is continuous).

For $S^1 = \{x \times y \mid x^2 + y^2 = 1\}$, we can write

$$g(x,y) = x^2 + y^2; \qquad S^1 = g^{-1}(\{1\}).$$

As $\{1\}$ ~~is~~ is closed and g is ~~con~~ continuous, S^1 is closed.

For $B^2 = \{x \times y \mid x^2 + y^2 \leq 1\}$, we can write

$$h(x,y) = x^2 + y^2; \qquad B^2 = h^{-1}([0,1]).$$

As $[0,1]$ is closed and h is continuous, B^2 is closed.

We can ~~proof~~ prove continuity in all cases by ~~co~~ constructing $f/g/h$ from multiplication and addition.

(11) Repetitive (from analysis).

(12) Repetitive (from analysis).

The Quotient Topology

~~Definition (Quotient Map): Let X and Y be topological spaces and let $p: X \to Y$ be a surjection. The map p is a quotient map if: a subset $U \subseteq Y$ is open if and only if p^{-1} its preimage under p, $p^{-1}(U)$, is open in X. Equivalently, p is a quotient map if the preimage of every closed set is closed. The condition is sometimes called "strong "strong continuity."~~

Definition (Saturated Set): Let X and Y be topological spaces and let $p: X \to Y$ be a surjection. A subset $C \subseteq X$ is <u>saturated</u> if $p^{-1}(\{y\}) \subseteq C$ for all $y \in Y$ with ~~$p^{-1}(\{y\}) \cap C$~~ $p^{-1}(\{y\}) \cap C \neq \emptyset$. That is, C contains every set $p^{-1}(\{y\})$ that it intersects. That is, C is saturated if there exists a $D \subseteq Y$ with $C = p^{-1}(D)$.

Definition (Quotient Map): Let X and Y be topological spaces and let $p: X \to Y$ be a surjection. Then the map p is a <u>quotient map</u> if any of the following equivalent definitions hold:

- a set $U \subseteq Y$ is open if and only if $p^{-1}(U)$ is open in X;
- a set $C \subseteq Y$ is closed if and only if $p^{-1}(C)$ is closed;
- p is continuous and maps saturated open sets $U \subseteq X$ to ~~pen~~ open sets $p(U) \subseteq Y$;
- p is continuous and maps saturated closed sets to closed sets.

Definition (Open/Closed Map): A map $f: X \to Y$ is <u>open</u> if for each open $U \subseteq X$, the image $f(U)$ is open. It is called <u>closed</u> if for each closed $C \subseteq X$, $f(C)$ is closed.

Proposition: A surjective ~~con~~ continuous map $p: X \to Y$ that is open or closed, is a quotient map.

Definition (Quotient Topology): Let X be a topological space and let A be a <u>set</u>. Let $p: X \to A$ be a surjection. Then there is exactly one topology $\mathcal{T}$ over A relative to which p is a quotient map; it is called the <u>quotient topology</u> induced by p. It contains exactly those subsets $U \subseteq A$ such that $p^{-1}(U)$ is open in X.

Definition (Quotient Space): Let X be a topological space and let X^* be a partition of X. Let $p: X \to X^*$ be the surjective map such that for all $x \in X$, $x \in p(x)$, i.e., it carries ~~one~~ each element of X to the element of X^* containing it. In the quotient topology induced by p over X^*, X^* is called ~~then~~ <u>quotient space</u> of X. Said differently, a set $U \subseteq X^*$ is a collection of equivalence classes and $p^{-1}(U)$ is just their union such that U is open iff the union of all equivalence classes is open in X.

Theorem (Subspace and Quotient Map): Let $p: X \to Y$ be a quotient map, let $A \subseteq X$ be a subspace saturated w.r.t. p, let $q: A \to p(A)$ be the map obtained by restricting p. Then:

(i) If A is open or closed, then q is a quotient map.

(ii) If p is an open or a closed map, then q is a quotient map.

~~Theorem (Maps out of Quotient Topolo~~

Theorem (Continuity and Quotient Maps): Let $p: X \to Y$ be a quotient map, let Z be a topological space, and let $g: X \to Z$ be a map such that it is constant on ~~on~~ each set $p^{-1}(\{y\})$, $y \in Y$. Then g induces a map $f: Y \to Z$ such that $f \circ p = g$. The induced map f is continuous if and only if g is continuous; f is a quotient map if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

Corollary (Maps out of Quotient Spaces): Let $g: X \to Z$ be a surjective continuous map. Let $X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$ and let X^* be equipped with the quotient topology. Then:

(i) The map g induces a bijective continuous map $f: X^* \to Z$ which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* \arrow[r, "f"'] & Z \end{tikzcd}$$

(ii) If Z is Hausdorff, so is X^*.

Exercises:

(1) Let $A = \{a, b, c\}$ and $p: \mathbb{R} \to A$ be defined by

$$p(x) = \begin{cases} a & \text{if } x > 0, \\ b & \text{if } x < 0, \\ c & \text{if } x = 0. \end{cases}$$

We have $p^{-1}(\emptyset) = \emptyset$, $p^{-1}(A) = \mathbb{R}$, $p^{-1}(\{a\}) = (0, \infty)$, $p^{-1}(\{b\}) = (-\infty, 0)$, and $p^{-1}(\{a, b\}) = \mathbb{R} \setminus \{0\}$, which are all open. Conversely, we have $p^{-1}(\{c\}) = \{0\}$, $p^{-1}(\{a, c\}) = [0, \infty)$, and $p^{-1}(\{b, c\}) = (-\infty, 0]$, which are not open. Thus, we have the quotient topology:

$$\mathcal{T} = \{\emptyset, A, \{a\}, \{b\}, \{a, b\}\}$$

(2) (a) *Proof.* Let $p: X \to Y$ be continuous and suppose there is a continuous $f: Y \to X$ such that $p \circ f$ equals the identity map of Y. We show that p is a quotient ~~ma~~ map. We first show that p is surjective. Let $y \in Y$, then $p(f(y)) = y$, so there is an $x = f(y)$ such that $p(x) = y$. Thus, p is surjective. It is continuous, so for all open $U \subseteq Y$, $p^{-1}(U)$ is open. We show the converse: let $U \subseteq Y$ such that $p^{-1}(U)$ is open. As f is continuous, $f^{-1}(p^{-1}(U))$ is open. However, as $p \circ f$ is the identity, $f^{-1}(p^{-1}(U)) = U$, such that U is open. Hence, p is a quotient map. $\square$

(b) *Proof.* Let X be a topological space and let $A \subseteq X$ be a subspace. Let $r: X \to A$ be a retraction from X to A, i.e.,* $r(a) = a$ for all $a \in A$. Clearly, r is surjective. ~~Moreov~~ Moreover, it is open: let $U \subseteq X$ be open, then $r(U) = U \cap A$ as for each $x \in U$ with $x \in A$, we have $r(x) = x$. ~~Thus, by definition, $U \cap A$ is~~ As $U \cap A$ is open by definition, $r(U)$ is open. ~~We still need to show that r is co con tinuous. Let $U \subseteq A$ be open. Then there is an open~~ As r is continuous by assumption, it is a quotient map. $\square$

~~(3) Proof. Let $\pi_1: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection onto the first a coordinate and let A be defined as~~

$$\cancel{A = \{x \times y \mid x \geq 0\} \cup \{x \times y \mid y = 0\},}$$

~~treated as a subspace of $\mathbb{R} \times \mathbb{R}$. Let $q: A \to \mathbb{R}$ denote the restriction of π_1 to A. It is clearly injective as for all $x \in \mathbb{R}$, we have $x \times 0 \in q^{-1}(\{x\})$. surjective We now state that $U \subseteq A$ $U \subseteq \mathbb{R}$ is open iff $q^{-1}(U)$ is open. "If:" Let $U \subseteq \mathbb{R}$ such that $q^{-1}(U)$ is open. Let $x \in U$. Then $x \times 0 \in q^{-1}(\{x\})$. If $x \geq 0$, additionally We then have~~

$$\cancel{q^{-1}(\{x\}) = \begin{cases} \{x \times 0\} & \text{if } x < 0, \\ \{x \times 0\} \cup (\{x\} \times \mathbb{R}) & \text{if } x \geq 0. \end{cases}}$$

~~Both options are open sets. Thus, $q^{-1}(U) = \bigcup_{x \in U} q^{-1}(\{x\})$ is open, too. "Only if:" Let $U \subseteq \mathbb{R}$ be open~~

(3) TODO

* And r needs to be continuous.

~~(4) (a) Proof. Let $X = \mathbb{R}^2$ and define the equivalence relation ~ as follows (for all $x_0 \times y_0, x_1 \times y_1 \in X$):~~

$$\cancel{x_0 \times y_0 \sim x_1 \times y_1 \iff x_0 + y_0^2 = x_1 + y_1^2}$$

~~Let X^* be the the corresponding quotient space~~

(4) (a) Let $X = \mathbb{R}^2$ and define ~ as

$$x_0 \times y_0 \sim x_1 \sim y_1 \iff x_0 + y_0^2 = x_1 + y_1^2$$

for all $x_0 \times y_0, x_1 \times y_1 \in X$. Clearly, ~ is an equivalence relation. Let X^* be the corresponding subspace.

Claim: X^* is homeomorphic to $\mathbb{R}$ under the standard topology.

Proof. Define $g: X \to \mathbb{R}$ as $g(x \times y) = x + y^2$. Clearly, g is continuous and surjective: for $x \in \mathbb{R}$, we have $g(x \times 0) = x + 0^2 = x$. Moreover, it is ~~open~~ an open map as for all basis elements $(a,b) \times (c,d)$ of X, we have

$$g((a,b) \times (c,d)) = (a + c^2, b + d^2),$$

which is open in $\mathbb{R}$. Thus, g is a quotient map. ~~Moreove~~ Moreover, it induces the quotient space X^* on X, i.e., $X^* = \{g^{-1}(\{z\}) \mid z \in \mathbb{R}\}$. This is due to $x_0 \times y_0 \sim x_1 \times y_1$ iff $g(x_0 \times y_0) = g(x_1 \times y_1)$. Hence, there is a homeomorphism $f: X^* \to Z$ and thus X^* and $\mathbb{R}$ are homeomorphic. □

(b) Analogous, the quotient space $X/\sim$ is homeomorphic to the standard topology on $[0, \infty)$.

(5) Proof. Let $p: X \to Y$ be an open map, let $A \subseteq X$ be open and let $q: A \to p(A)$ be the restriction of p to A. Let $U \subseteq A$ be open in A. Then U is also open in X as A is open. Hence, $p(U)$ is open. As $U \subseteq A$, we have $\cancel{p(}\, q(U) = p(U)$, ~~so q(U) is open, too.~~ so $q(U)$ is open in X, too. As $q(U) \subseteq p(A)$, $q(U)$ is also open in $p(A)$, so q is an an open map. □

(6) Let $\mathbb{R}_K$ be the real line in the K-topology with basis all open intervals (a,b) and all $(a,b)\setminus K$ where $K=\{1/n \mid n\in\mathbb{N}\}$. Let Y be the quotient space obtained from $\mathbb{R}_K$ by identifying K to a point. Let $p:\mathbb{R}_K\to Y$ be the quotient map.

(a) Proof. We first show that Y satisfies the T_1 axiom. Let $y\in Y$. If $y=K$, then ~~$Y\setminus\{y\}$ contains $\{x\}$ for each $x\in\mathbb{R}$, i.e. $\bigcup_{x\in Y\setminus\{y\}}\{x\}=\mathbb{R}$. As $\mathbb{R}$ is open in $\mathbb{R}_K$, $\{y\}$ is closed.~~

$$Y\setminus\{y\}=\{\{x\}\mid x\in\mathbb{R},\ \cancel{x\neq 1/n}\ x\notin K\}.$$

We thus have $\bigcup(Y\setminus\{y\})=\mathbb{R}\setminus K$, denoting the union over all elements of $Y\setminus\{y\}$. ~~We can re-write this as $\mathbb{R}\setminus K=\mathbb{R}\cap(\mathbb{R}\setminus K)$ and as K is closed~~ As $\mathbb{R}\setminus K$ is open in $\mathbb{R}_K$, $\{y\}$ is closed. If $y\neq K$, then ~~$\mathbb{R}\setminus K=\{$~~ ~~Hence~~ there is ~~an~~ a $\tilde{y}\in\mathbb{R}$ with $y=\{\tilde{y}\}$. Taking the union over $Y\setminus\{y\}$, we get $\mathbb{R}\setminus\{\tilde{y}\}$, which is clearly open and thus $\{y\}$ is closed in Y. Hence, Y satisfies the T_1 axiom.

It is not ~~Haus~~ Hausdorff. Consider $\{0\}, K\in Y$. Clearly, $\{0\}\neq K$. Let $U,V\subseteq Y$ be open sets with $\{0\}\in U$ and $K\in V$. Then $p^{-1}(U)$ is open and there are $a,b\in\mathbb{R}$, ~~Clear $a<0$~~ $a<0<b$, such ~~tha~~ that $(a,b)\subseteq p^{-1}(U)$. But then, as $(a,b)\cap K\neq\emptyset$, also ~~(a,b)~~ $p^{-1}(U)\cap p^{-1}(V)\neq\emptyset$. However, $p^{-1}(U)\cap p^{-1}(V)=p^{-1}(U\cap V)$, so $U\cap V\neq\emptyset$. ~~Hence~~ Hence, as U and V were arbitrary, the space Y is not Hausdorff.

$\square$

(b) Proof. Let $p\times p:\mathbb{R}_K\times\mathbb{R}_K\to Y\times Y$ be defined by $(p\times p)(x\times y)=p(x)\times p(y)$. Consider the diagonal

$$A=\{y\times y\mid y\in Y\}.$$

~~Due to exercise 13 (§~~ Due to exercise 13 (§17), the diagonal is closed iff Y is Hausdorff. As Y is not Hausdorff, A is not closed. However, we have $(p\times p)^{-1}(A)=\Delta\cup K\times K$, where Δ is the diagonal in $\mathbb{R}_K\times\mathbb{R}_K$. Δ is closed. ~~K is~~ As K is closed in $\mathbb{R}_K$, $K\times K$ is closed, too. Thus, the preimage of A under $p\times p$ is closed. Hence, it is not a ~~quotient~~ quotient mapping by definition.

$\square$

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Topological ~~Groups~~ Groups

Definition (Group): A group $(G, \cdot)$ is a set G equipped with an operation $\cdot : G \times G \to G$, denoted $a \cdot b$ for all $a, b \in G$, such that the following group axioms are satisfied:

(i) for all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity);

(ii) there is an element $e \in G$, ~~often denoted by 1, such~~ that for all $a \in G$, $e \cdot a = a \cdot e = a$ (identity element);

(iii) for all $a \in G$, there is an element ~~b~~ $b \in G$, often denoted a^{-1}, such that $a \cdot a^{-1} = a^{-1} \cdot a = e$, where e is the identity element (inverse element).

Definition (Topological Group): A topological group G is a group that is also a topological space satisfying the T_1 axiom, such that ~~them~~ the maps $x \times y \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are continuous.

(1) Proof. Let G be a ~~topolo~~ ~~g~~ group that is also a topological space satisfying the T_1 axiom. ~~"If:" Suppose that the map $x \times y \mapsto x \cdot y^{-1}$ is continuous.~~ Let $h : G \to G$, ~~and~~ $g : G \times G \to G$, and $f : G \times G \to G$ be the maps $h(x) = x^{-1}$, $g(x, y) = x \cdot y$, and $f(x, y) = x \cdot y^{-1}$, respectively. "If:" Suppose that f is continuous. Define $i : G \to G \times G$ by $i(x) = e \times x$, where ~~e is the~~ $e \in G$ is the identity. Then i is continuous and we have $h = f \circ i$, so h is continuous. Define $j : G \times G \to G \times G$ by $j(x, y) = x \times h(y)$, then $g = f \circ j$. We show that j is continuous in each coordinate. Clearly, $j_1(x, y) = x$ is continuous as for all open $U \subseteq G$, $j_1^{-1}(U) = U \times G$. Moreover, $j_2(x, y) = h(y)$ is continuous as for all open $U \subseteq G$, we have $j_2^{-1}(U) = G \times h^{-1}(U)$ where $h^{-1}(U)$ is open as h is continuous. Thus g is continuous and ~~that~~ therefore G is a topological ~~g~~ group. "Only if:" Suppose h and g are continuous. Let $j(x, y) = x \times h(y)$, then $f = g \circ j$. As j is continuous (c.f. above), f is. $\square$

(2) (a) Proof. It is clear that $(\mathbb{Z}, +)$ is a group with identity 0 and inverse $-x$ for all $x \in \mathbb{Z}$. Endowed with the order topology, it satisfies the T_1 axiom. Let (a, b) be a typical basis element. Then its preimage under the map $x \mapsto -x$ is $(-b, -a)$, which is open, so the map is continuous. Continuity of $x \times y \mapsto x + y$ is easily seen by restricting addition on the reals to $\mathbb{Z}$. $\square$

(b) See (a).

(2) (c) Proof. Clearly, $(\mathbb{R}_+, \cdot)$ is a group and with the usual topology, it is Hausdorff and thus satisfies the T_1 axiom. ~~Let $f(x\times y)=xy$ and $h(x)=1/x$ denote the 2 maps x~~ Let $f: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ and $h: \mathbb{R}_+ \to \mathbb{R}_+$ denote the maps $x \times y \mapsto xy$ and $x \mapsto 1/x$, respectively. ~~As~~ By restricting addition and division from the reals to $\mathbb{R}_+$, we see that both f and h are continuous.

$\square$

(d) Proof. Clearly, S^1 is a group. As S^1 is homeomorphic to $[0,1) \subseteq \mathbb{R}$, it is Hausdorff and the maps $x \times y \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are continuous.

$\square$

(e) Clearly a group, homeomorphic to $\mathbb{R}^{n^2}$ by contraction.

(3) Proof. Let G be a topological ~~gr~~ group and let H be a subspace of G. Suppose that H is also a subgroup of G. H satisfies the T_1 ~~axiom~~ axiom as G does and the maps $x \times y \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are continuous by appropriately restricting the domain and range. For $\bar{H}$, ~~not~~ we have to ~~show~~ show that it is a group, that is, that $x \times y \mapsto x \cdot y$ and $x \mapsto x^{-1}$ are closed in $\bar{H}$. Everything else will follow readily. First for $x \mapsto x^{-1}$. Let $x \in \bar{H}$. We show that also $x^{-1} \in \bar{H}$. Let $U \subseteq G$ be a neighborhood of x^{-1} and denote by U^{-1} the preimage of U under $x \mapsto x^{-1}$. Then, as $x^{-1} \in U$ and $(x^{-1})^{-1} = x$, we have $x \in U^{-1}$. Moreover, as U is open and the map is continuous, U^{-1} is open. Thus, as $x \in \bar{H}$, we have $U^{-1} \cap H \neq \emptyset$. But noting that $x \mapsto x^{-1}$ is bijective, we have $(U^{-1} \cap H)^{-1} = (U^{-1})^{-1} \cap H^{-1} = U \cap H$ ~~$\neq \emptyset$~~ as H is a group, so $H^{-1} = H$. Hence, $x^{-1} \in \bar{H}$ as $U \cap H \neq \emptyset$. Similar results follow for $x \times y \mapsto x \cdot y$.

$\square$

(4) Proof. Let G be a topological group, fix $\alpha \in G$ and define $f_\alpha, g_\alpha: G \to G$ as $f_\alpha(x) = \alpha \cdot x$ and $g_\alpha(x) = x \cdot \alpha$. We show that f_α is a homeomorphism. It is clearly bijective as for $f_\alpha^{-1}(y) = \alpha^{-1} \cdot y$, we have

$$f_\alpha^{-1}(f_\alpha(x)) = \alpha^{-1} \cdot \alpha \cdot x = x \quad \text{and}$$

$$f_\alpha(f_\alpha^{-1}(x)) = \alpha \cdot \alpha^{-1} \cdot x = x.$$

Moreover, let ~~$i(x)=$~~ $i_\beta(x) = \beta \times x$. Then $f_\alpha = f \circ i_\alpha$ and $f_\alpha^{-1} = f \circ i_{\alpha^{-1}}$ where ~~$f(x)=$~~ $f(x \times y) = x \cdot y$, so f_α is a homeomorphism. g_α is analogous. Hence, G is homogeneous as for all $x, y \in G$, we have the homeomorphism f_α with $\alpha = y \cdot x^{-1}$ and $f_\alpha(x) = y \cdot x^{-1} \cdot x = y$.

$\square$

(5) Let G be a topological group, let ~~H~~ H be a subgroup of G and define, for all $x \in G$, $xH = \{x \cdot h \mid h \in H\}$ (the ~~x~~ <u>left coset</u> of H in G). Let ~~H\G~~ G/H be the collection of left cosets, i.e., $G/H = \{xH \mid x \in G\}$. Equip G/H with the quotient topology (note that G/H partitions G).

(a) Proof. Fix $\alpha \in G$ and define $f_\alpha : G/H \to G/H$ as

$$\cancel{f_\alpha(xH) = (x \cdot \alpha)H,} \qquad f_\alpha(xH) = (\alpha \cdot x)H$$

Clearly, f_α is bijective as $f_{\alpha^{-1}}$ is its inverse. ~~We show that f_α is continuous, continuity of $f_{\alpha^{-1}}$ follows by symmetry. Let $U \subseteq G/H$ be open. That is, the union $\bigcup_{xH \in U} xH$ is open. We then have~~

$$\cancel{f_\alpha^{-1}(U) = f_{\alpha^{-1}}(U) = \{(\alpha^{-1} \cdot x)H \mid xH \in U\}}.$$

~~Taking the union over the elements of the RHS~~,

$$\cancel{p^{-1}(f_\alpha^{-1}(U)) = \bigcup_{xH \in U} (\alpha^{-1} \cdot x)H}$$

~~We show that f_α is open, openness and thus continuity of $f_{\alpha^{-1}}$ and~~ We show that f_α is open. Continuity of f_α, and openness and continuity of $f_\alpha^{-1} = f_{\alpha^{-1}}$ follow immediately. Let $U \subseteq G/H$ be open. Then, by definition,

$$U = \{xH \mid x \in p^{-1}(U)\},$$

where $p : G \to G/H$ is the quotient map. Then

$$f_\alpha(U) = \{(\alpha \cdot x)H \mid x \in p^{-1}(U)\}$$
$$\cancel{= \{\alpha y H \mid} \qquad = \{yH \mid y \in \alpha p^{-1}(U)\},$$

where $\alpha p^{-1}(U) = \{\alpha \cdot x \mid x \in p^{-1}(U)\}$. As the map $x \mapsto \alpha \cdot x$ is a homeomorphism, $\alpha p^{-1}(U)$ is open such that $f_\alpha(U)$ is open. Hence, f_α is a homeomorphism and G/H is a homogeneous space. $\square$

(b) Proof. Suppose H is closed in G. Let $xH \in G/H$ and consider $\{xH\}$ in G/H. Then $p^{-1}(\{xH\}) = xH$. As H is closed in G and ~~$x \mapsto x \cdot a$ is~~ $y \mapsto x \cdot y$ is homeomorphic, xH is closed. Hence, $\{xH\}$ is closed in ~~H~~ G/H. $\square$

. . . .

Claim: Every subspace of a topological space satisfying the T_1 ~~a~~ axiom satisfies the T_1 axiom.

Proof. Let X be a topological space satisfying the T_1 ~~a~~ axiom. Let $Y \subseteq X$ be a ~~a~~ subspace. Let $y \in Y$ and consider the set $\{y\}$. Then $\{y\}$ is closed in X and as $\{y\} = \{y\} \cap Y$, it is also closed in Y. $\square$

. . . .

~~(5) (c) Proof. Let $p: G \to G/H$ be the quotient map.~~

(5) (c) Proof. Let $p: G \to G/H$ be the quotient map. Let $U \subseteq G$ be open. Then

$$p(U) = \{ xH \mid x \in U \}$$

and

$$p^{-1}(p(U)) = \bigcup_{x \in U} xH.$$

Let $x' \in \bigcup_{x \in U} xH$. Then there is an $\alpha \in H$ such that $x' = x \cdot \alpha$.

TODO

(d) TODO

(6) Consider $\mathbb{Z}$ as a subgroup of $(\mathbb{R}, +)$. Then $\mathbb{R}/\mathbb{Z}$ is homeomorphic to ~~[0,1]~~ $[0,1]$, but with what operation?

TODO

~~(7) (a) Proof. Let G be a topological group with with identity element $e \in G$. Let $U \subseteq G$ be open with $e \in U$.~~

(7) TODO